Lecture 11
Waves in capillary plasmas and capillary Z-pinch plasmas

Outline

• Development of plasma waveguides for an electron acceleration using the capillary z-pinches developed at University of Pecs. The light confinement is limited by diffraction. Development of plasma waveguides for electron acceleration

• Development of plasma waveguides: Phase matching by using modal properties of capillary z-pinch waveguides. MHD for plasma based waveguides: 2 temperatures ($T_e$, $T_i$) and 2 fluids ($n_e$, $V_e$, $n_i$, $V_i$).

• Basic properties of TEM waves in a plasma. Propagating and damping TE-waves in a plasma. The cut-off (critical) frequency and e-density. Phase velocity vs group velocity of TEM waves. Attenuation of a TEM wave in plasma by electron-ion collisions. Linear and nonlinear EM waves in plasma.

• Modal properties of a laser beam guided by a gradient-refraction-index plasma column. Summary: Guiding of a laser beam (wave, rays) by a gradient-refraction-index plasma column. Guiding of beam rays by a plasma-based waveguide: Transient modes ($m=0,1,2 \ldots$). Guiding of a beam wave by a plasma-based waveguide: Transient modes $m$ ($m=0,1,2 \ldots$). The transient modes produced by reflection of the plasma leaky-mode ($m$) from the capillary walls.
• A waveguide by capillary Z-pinch discharge. Generation of a long (b >> b_R), gradient-index, plasma-waveguide by a capillary Z-pinch. 1-D MHD for a Z-pinch plasma waveguide: 2 temperatures (T_e, T_i) and 2 fluids (n_e, V_e, n_i, V_i). 1-D MHD for a Z-pinch plasma waveguide: 2 temperature (T_e, T_i) and 1 fluid (ρ, V_r)
• Set-up for plasma waveguides for electron acceleration based on Z-pinch at UP.
• Understanding waves in Z-pinch requires theory, computations and experiments
• Problems as home assignments
• References
Development of plasma waveguides for electron acceleration using the capillary Z-pinches developed at University of Pécs

Let me briefly consider the topic, which is closely related to the R&D of capillary x-ray lasers at the University of Pécs. The detailed theory of acceleration is beyond the scope of the lecture devoted to x-ray lasers.

**Plasma-wakefield acceleration of particles by an intensive laser pulse:**

Principle of LWA using an electric field associated with electron plasma wave (like a shock or sheath field)

LWA in a capillary hydrogen plasma (h~1%) with $L \sim 5 \text{ cm}$ ($U_{\text{max}} \sim 20 \text{ kV}$, $I_{\text{max}} \sim 1 \text{ kA}$):
www-alphys.physics.ox.ac.uk

**Fig. 1** Plasma-wakefield acceleration of particles by an intensive laser pulse

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Light confinement is limited by diffraction

The techniques offer a way to build particle accelerators of much smaller size than conventional devices. Lawrence Berkeley National Laboratory accelerates electrons to 1 GeV over $L = 3.3 \, \text{cm}$

$$E \text{ up to} \ 1 \, \text{GV/cm} \ \rightarrow \ 50 \, \text{GeV} \ \text{electrons with using the plasma column} \ L = 50 \, \text{cm}$$

Fig. 2 Light confinement is limited by diffraction.

Light confinement is limited by diffraction to the Rayleigh length:

$$b_R = \frac{\pi r_0^2}{\lambda}$$
Development of plasma waveguides for electron acceleration

Without guiding (for example, without self-focusing or preformed channels), the laser–plasma interaction length is limited to the order of the Rayleigh range, $Z_R$, (a few millimetres for $r_{spot} = 25 \, \mu m$). Relativistic self-guiding can extend the propagation distance of high-power pulses due to self-consistent modification of the plasma refractive index, but is limited by nonlinear effects.

A more efficient approach relies on channelling laser beams with smaller spot sizes over centimetre-scale distances. Theory and simulation indicate that such channel-guided accelerators could produce GeV e-beams with 10–50 TW of laser power\(^2\),\(^3\),\(^4\),\(^14\). However, simply making the accelerator longer is not sufficient. Phase slippage occurs between relativistic particles and the wake, because the wake has a phase velocity less than the vacuum speed of light. The linear dephasing length, $L_d = \frac{\lambda_p^3}{\lambda^2} \propto n_p^{-3/2}$, over which electrons outrun the wake and slip into the decelerating phase, limits the distance over which acceleration occurs. Here $\lambda_p$ is the plasma wavelength, $\lambda$ is the laser wavelength and $n_p$ is the plasma density. For laser intensities $I \leq 10^{18} \, \text{W} \, \text{cm}^{-2}$, a rough estimate of the electron energy gain over a distance $L_d$ in a channel-guided laser-wakefield accelerator\(^2\),\(^14\) can be obtained from $W \, (\text{GeV}) \approx 0.4 I \, (\text{Wcm}^{-2}) / n_p \, (\text{cm}^{-3}) \approx 0.9 (\lambda_p / r_s)^2 P \, (\text{TW})$; where $P$ is the laser peak power (in TW). In the 2004 experiments, matching acceleration length to $L_d$ led to the production of low-energy-spread e-beams\(^3\),\(^5\).
Development of plasma waveguides: Phase matching by using modal properties of capillary Z-pinch waveguides
(25-year experience of University of Pécs in the waveguides)

Fig. 5 Schema of the experimental set-up for R&D of plasma waveguides. The phase matching required for the electron acceleration would be provided by the different physical (phase) properties of the waveguide modes.

Let me briefly consider the topic, which is closely related to the R&D of capillary x-ray lasers at the University of Pécs. The detailed theory of acceleration is beyond the scope of the lecture devoted to x-ray lasers.

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MHD for a plasma waveguides: 2 temperatures \((T_e, T_i)\) and 2 fluids \((n_e, V_e, n_i, V_i)\)

\[
\begin{align*}
\frac{dn_e}{dt} + \text{div}(n_e V_e) &= 0 \quad (1) \\
\frac{dn_i}{dt} + \text{div}(n_i V_i) &= 0 \quad (2)
\end{align*}
\]

\[
\begin{align*}
m_e n_e \frac{dV_{e\alpha}}{dt} &= dP_{e\alpha}/dx_{\alpha} - d\pi_{e\alpha\beta}/dx_{\beta} - En_e(E_{\alpha} + (V_e \times B)_{\alpha}) + R_{\alpha} \quad (3) \\
m_i n_i \frac{dV_{i\alpha}}{dt} &= dP_{i\alpha}/dx_{\alpha} - d\pi_{i\alpha\beta}/dx_{\beta} + Zen_i(E_{\alpha} + (V_i \times B)_{\alpha}) - R_{\alpha} \quad (4)
\end{align*}
\]

\[
\begin{align*}
(3/2)n_e \frac{d}{dt}T_e &= -P_e \text{div} V_e - \pi_{e\alpha\beta} dV_{e\alpha}/dx_{\beta} - \text{div} Q_e + W_e \quad (5) \\
(3/2)n_i \frac{d}{dt}T_i &= -P_i \text{div} V_i - \pi_{i\alpha\beta} dV_{i\alpha}/dx_{\beta} - \text{div} Q_i + W_i \quad (6)
\end{align*}
\]

\[
\begin{align*}
P_e &= n_e T_e, \quad P_i = n_i T_i \quad (10)
\end{align*}
\]

For the adiabatic stage \((Q = 0)\)

\[
\begin{align*}
P_e/P_{0e} &= (n_e/n_{0e})^{5/3} \quad (8) \\
P_i/P_{0i} &= (n_i/n_{0i})^{5/3} \quad (9)
\end{align*}
\]

For a plasma waveguide, the plasma viscosity was taken into account

\[
\text{For details, see S.I. Braginskii, Transport processes in a plasma, Reviews of Plasma Physics, 1, AP. NY (1965)).}
\]

Maxwell equations

\[
\begin{align*}
\frac{d}{dt} = d / dt + V_e (d/dr) \quad (11) \\
\frac{d}{dt} = d / dt + V_i (d/dr) \quad (12)
\end{align*}
\]
Basic properties of TEM waves in a plasma

Let us consider the transverse EM waves in a plasma. In the case of the transverse EM waves produced by the transverse oscillation of electrons, Maxwell equations and MHD equations for a viscosity-less plasma yield

\[
\nabla \times E_T = -\mu_0 \frac{\partial H_T}{\partial t} \tag{14}
\]

\[
\nabla \times H_T = \varepsilon_0 \frac{\partial E_T}{\partial t} - e n_e v_T \tag{15}
\]

\[
m \frac{d e v_T}{dt} = \frac{1}{n_e} \nabla P - \frac{1}{n_e} e n_e (E_T + v_T \times B) \approx -eE_T \tag{16}
\]

Taking carle of Eq. (1) and \(\partial / \partial t\) of Eq. (2) yields

\[
\nabla \times \nabla \times E_T = -\mu_0 \nabla \times \frac{\partial H_T}{\partial t} \tag{17}
\]

\[
\nabla \times \frac{\partial H_T}{\partial t} = \varepsilon_0 \frac{\partial^2 E_T}{\partial t^2} - e n_e \frac{\partial^2 v_T}{\partial t^2} \tag{18}
\]

Then the use of the equality

\[
\nabla \times \nabla \times E_T = \nabla (\nabla E_T) - \nabla^2 E_T = -\nabla^2 E_T \tag{19}
\]

we get

\[
\nabla^2 E_T = -\varepsilon_0 \mu_0 \frac{\partial^2 E_T}{\partial t^2} + \mu_0 e n_e \frac{\partial v_T}{\partial t} \tag{20}
\]

\[
\frac{\partial^2 E_T}{\partial t^2} + \frac{1}{\varepsilon_0 \mu_0} \nabla^2 E_T - \frac{e n_e}{\varepsilon_0} \frac{\partial v_T}{\partial t} = 0 \tag{21}
\]

\[
\frac{\partial^2 E_T}{\partial t^2} + \frac{1}{\varepsilon_0 \mu_0} \nabla^2 E_T + \frac{e^2 n_e}{\varepsilon_0 m} E_T = 0 \tag{22}
\]

For the plane TE wave, Eq. (10) yields

\[
\omega^2 = \omega_p^2 - k^2 \tag{24}
\]
Propagating and damping TE-waves in a plasma

Thus we have the wave equation

$$\left( \frac{\partial^2}{\partial t^2} + c^2 \nabla^2 + \omega_p^2 \right) E_T = 0$$

(25)

which for the plane TE wave

$$E_T(r, t) = E_0 e^{i(kr - \omega t)}$$

(26)

yields the dispersion relation

$$\omega^2 = \omega_p^2 - k^2$$

(27)

The solution of Eq. (14) for k yields

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}$$

(28)

In the case of under-dense plasma ($n_e < n_c = \varepsilon_0 m \omega^2/e^2$), the wave number is given by

$$k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c} > 0$$

(29)

which yields the propagating wave (13). In the case of over-dense ($n_e > n_c$) plasma,

we have the damping wave

$$E_T(r, t) = E_0 e^{-|k|r} e^{i\omega t}$$

(31)

with the penetration length ($\omega << \omega_p$)

$$\ell \sim \frac{c}{\omega_p}$$

(32)
Cut-off (critical) frequency and e-density. Plasma refraction index.

Wave frequency \( \omega = \omega_p \)

Where plasma frequency \( \omega_p \) is given by

\[
\omega_p = \sqrt{\frac{e^2 n_e}{\varepsilon_0 m}}
\]

(34)

is called a cut-off or critical frequency. According to Eq. (33) and (34), the critical frequency corresponds to the critical electron density as

\[
n_c = \frac{\varepsilon_0 m \omega_p^2}{e^2}
\]

(35)

or in terms of the wavelength in nm

\[
n_c = \frac{1.1 \times 10^{21} e / cm^2}{\lambda^2 (\mu m)}
\]

(36)

For instance, the critical electron density for wavelength \( \lambda = 1 \mu m \) has the value \( n_c \sim 10^{-21} (e/cm^3) \).

In the case of \( \omega > \omega_p \) the wave \( E_1(r, t) = E_c e^{i(kr - \omega t)} \)

(37)

propagates with phase velocity

\[
v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{c}{\sqrt{1 - \frac{n_e}{n_c}}}
\]

(38)

Using relation \( v_{ph} = c/n \), the plasma refraction index can be presented as

\[
n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \sqrt{1 - \frac{n_e}{n_c}}
\]

(39)
The phase and group velocities of an EM wave are determined by the dispersion relation

$$\omega^2 = \omega_p^2 - k^2$$  \hspace{1cm} (40)

respectively as

$$v_{ph} = \frac{\omega}{k}$$  \hspace{1cm} (41)

$$v_g = \frac{\partial \omega}{\partial k}$$  \hspace{1cm} (42)

Then, Eqs. (40) - (42) yield

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} = \frac{c}{\sqrt{1 - \frac{n_e}{n_c}}}$$  \hspace{1cm} (43)

$$v_g = \frac{\omega}{k} = c \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = c \sqrt{1 - \frac{n_e}{n_c}}$$  \hspace{1cm} (44)

For $n_e \ll n_c$, we have

$$v_{ph} \approx v_g$$  \hspace{1cm} (45)

while the relation $n_e \to n_c$,

$$v_{ph} \to \infty$$  \hspace{1cm} (46)

$$v_g \to 0$$  \hspace{1cm} (47)
Attenuation of TEM wave in plasma by electron-ion collisions

In the case of the high value of electron-ion collision frequency $\nu_{ei}$, the Newton equation of motion has the form

$$m \frac{\partial \mathbf{v}_T}{\partial t} = -e \mathbf{E} - m \nu_{ei} \mathbf{v}_T$$

(48)

which yields the velocity

$$\mathbf{v} = -\frac{ie}{m(\omega + i \nu_{ei})} \mathbf{E}$$

(49)

Then, for $\omega >> \nu_{ei}$, the dispersion relation modifies to

$$\omega^2 = \omega_p^2 \left(1 - i \frac{\nu_{ei}}{\omega}\right) - k^2$$

(50)

Presenting the frequency $\omega$ as $\omega = \omega_r + i\omega_{im}$, we find

$$\omega_r^2 = \omega_p^2 - k^2$$

(51)

$$\omega_{im} \approx -\frac{\nu_{ei}}{2} \left(\frac{\omega_p}{\omega}\right)^2 = -\frac{n_e}{2n_c} \nu_{ei}$$

(52)

Thus we have the damping wave

$$\mathbf{E}_T(r, t) = E_0 e^{-\omega_{im} t} e^{i k \mathbf{r} \cdot \omega_r t}$$

(53)
Linear and nonlinear EM waves in plasma

A simple analysis shows that the terms in boxes of the Maxwell equations

\[ \nabla \times \mathbf{E}_T = -\mu_0 \frac{\partial \mathbf{H}_T}{\partial t} \quad (54) \]

\[ \nabla \times \mathbf{H}_T = \varepsilon_0 \frac{\partial \mathbf{E}_T}{\partial t} - e n_e \mathbf{V}_T \quad (55) \]

and MHD equations (see, Lecture (6) and Eqs. (37) and (39))

\[ \frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{v}) = 0 \quad (56) \]

\[ m_j n_j \frac{d \mathbf{v}_\alpha}{dt} = \frac{d P_{\alpha}}{dx_{\alpha}} + q_j n_j \left( \mathbf{E}_\alpha + (\mathbf{V}_e \times \mathbf{B})_\alpha \right) \quad (57) \]

lead to nonlinear growth and frequency mixing. Mixing waves should satisfy the energy and momentum conservation for both linear and nonlinear mixing Processes. For instance, three-wave mixing yields the current densities

\[ J_{\alpha} e^{i (\mathbf{k}_\alpha \cdot \mathbf{r} - \omega_t t)} = -e n_e e^{i (\mathbf{k}_\alpha \cdot \mathbf{r} - \omega_t t)} v e^{i (\mathbf{k}_\alpha \cdot \mathbf{r} - \omega_t t)} \quad (58) \]

Satisfying conservation conditions

\[ \omega_1 = \omega_2 \pm \omega_3 \quad (59) \]

\[ \mathbf{k}_1 = \mathbf{k}_2 \pm \mathbf{k}_3 \quad (60) \]
Modal properties of laser beam guided by gradient-refraction-index plasma column

The channeling and guiding are the guide length an wavelength dependent [5]. That is different from the eigenmode model of H.M. Miltchberg et al. [6], where the channeling and guiding conditions are independent from the guide length and wavelength.

Fig. 6 Guiding and “focusing” of laser beam by gradient-index plasma guide (from [5]: S.V. Kukhlevsky, L. Kozma, Contributions to Plasma Physics 38: 583-597 (1998)
Summary: Guiding of laser beam (wave, ray) by gradient-refraction-index plasma column

Within MHD approximation, WE yields
\[
\left( \frac{\partial^2}{\partial z^2} + \nabla_\perp^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \widetilde{E} = \frac{\omega_p^2}{c^2} \cdot \frac{N_e(r)}{N_e(0)} \widetilde{E} + \vec{\nabla}(\vec{\nabla} \cdot \widetilde{E})
\] (64)

The ray-tracing approximation of Eq. (64) yields
\[
\frac{d}{ds} \left( n \frac{dr}{ds} \right) = \vec{\nabla} n
\] (65)

The paraxial-envelope approximation of Eq. (64) yields
\[
2ik \frac{\partial E}{\partial \xi} = \nabla_\perp^2 E + \frac{\omega_p^2}{c^2} \left( \frac{N_e(r)}{N(0)} - 1 \right) \mathcal{E}
\] (66)

From Eq. (65), we get ray trajectories in the paraxial approximation (small angles) with:
\[
\frac{r}{C/\gamma} = \frac{r(0)}{C/\gamma} \cos \left( \frac{z}{C/\gamma} \right) + \frac{1}{n(r)} \cdot \frac{dr(0)}{dz} \sin \left( \frac{z}{C/\gamma} \right)
\] (67)

From Eq. (67), we see that beam intensity distribution is periodically reproduced with the period \( z_p \):
\[ P(r, z = 0, \varphi) = P(r, z + m z_p, \varphi) \] (69)

The plasma guide of length \( L \) supports transient mode \( m \) \((m = 0, 1, 2, \ldots)\) if
\[
\frac{L}{z_p} = \frac{L}{\sqrt{2\pi} C} \cdot \sqrt{\frac{N_e(C) - N_e(0)}{(2\pi/r_e \lambda^2) - N_e(0)}} = m
\] (70)

The plasma does guide x-rays if the value \( L/z_p > 0.1 \), which yields the condition
\[
\Delta N_e^{\text{min}} = \frac{0.2\pi^2 C^2}{L^2} \cdot [2\pi/r_e \lambda^2 - N_e(0)]
\] (71)

(Transient modes in a plasma waveguide were introduced by S.V. Kukhlevsky et al. in Contr. Plasma Phys. 38: 583-597 (1998))
Guiding of beam rays by plasma-based waveguide: Transient modes $m$ ($m=0,1,2 \ldots$)

$$m = \frac{L}{z_p} \sim \frac{h^{1/2}}{L/x} \left[\frac{(2^{1/2})\pi C}{C}\right]^{-1}$$

Quality

$$h = \frac{N_e(r_a) - N_e(0)}{N_e(r_w)}$$

Fig. 7 Guiding and focusing of rays by plasma guide. Transient modes ($m$).

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Guiding of beam wave by plasma-based waveguide: Transient modes $m (m=0,1,2,\ldots)$

$$2ik \frac{\partial E}{\partial \zeta} = \nabla_{\perp}^2 E + \frac{\omega_p^2}{c^2} \left( \frac{N_e(r)}{N(0)} - 1 \right) E$$

solutions

$$\bar{E} = e_0 E(r, z) \exp(-ik\phi(z))$$

$$E^2(r, z) = E_0^2 \frac{w_0^2}{w^2(z)} \exp[-2\left(\frac{r}{w(z)}\right)^2]$$

$$\frac{d^2 W}{dz^2} = \frac{1}{k^2 w_0^2} \cdot \frac{1}{W^3} - \left(\frac{N_e(C) - N_e(0)}{N_c}\right) \cdot \left(\frac{w_0}{C}\right)^2 W$$

$$\frac{d\phi}{dz} = -\frac{1}{k^2 w_0^2} \cdot \frac{1}{W^2}$$

$$w_0 = \left(\frac{C^2}{4\pi r_e} \cdot \frac{1}{N_e(C) - N_e(0)}\right)^{1/4}$$

The normalised size of the central part of the beam $r_c/0.1w_0$ as function of the normalized propagation distance $z/w_0$: (A, B) vacuum diffraction, where $\Pi = 0$; (C, D) a channel-guided beam, without diffraction $\Gamma = 0$. The beam and plasma parameters are given by: (A) $\Gamma = 0.001$ and $W(\Delta_1) = 0$; (B) $\Gamma = 0.001$ and $W(\Delta_1) = 0.05$; (C) $\Pi = 0.01$ and $W(\Delta_1) = 0$; (D) $\Pi = 0.01$ and $W(\Delta_1) = 0.05$.

Fig. 8 Guiding and focusing of waves by plasma guide. Transient modes $m$. (from S.V. Kukhlevsky et al., Contr. Plasma Phys. 38: 583-597 (1998))
Transient modes produced by reflection of plasma leaky-mode \((m)\) from the capillary walls

Fig. 9 A The transient modes \(E_m\) produced by reflection of leaky plasma modes \((m)\) from the capillary walls (Figures from S.V. Kukhlevsky and G. Nyitray, Phys. Let. A, 291, 459 (2001))
Waveguide by capillary Z-pinch discharge

Fig. 10 Waveguide by Z-pinch, capillary Z-pinch or capillary discharge

Waveguide quality

\[ m = \frac{L}{z_p} \sim (\frac{h^{1/2}}{L}) \times [(2^{1/2})\pi C]^{-1} \] - Number of waveguide modes

\[ N_e (r') = N_e (0) + \Delta N_e \left( \frac{r}{r_{ch}} \right)^2 \]

\[ n(r) = \left[ 1 - \left( \frac{N_e (r)}{N_c} \right) \right]^{1/2} \]

\[ N_c = \pi m_e c^2 / e^2 \lambda^2 \]
Generation of long \((b >> b_R)\), gradient-index, plasma-waveguide by capillary Z-pinch

\[
\Delta \varphi = \frac{2\pi}{\lambda} \int_{r_1}^{r_2} (n-1) dl = \frac{2\pi}{\lambda} \int_{r_1}^{r_2} \left[ \left(1 - \frac{n_e}{n_c}\right)^{1/2} - 1 \right] dl
\]

\[
N_e(r) = N_e(0) + \Delta N_e \left( \frac{r}{r_{ch}} \right)^2
\]

\[
n(r) = [1 - (N_e(r)/N_C)]^{1/2} , \text{ where } N_C = \frac{\pi m_e c^2}{e^2 \lambda^2}
\]

\[
m = \frac{L}{z_p} \sim (h^{1/2} L) \times [(2^{1/2})\pi C]^{-1}
\]

\[
h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} - \text{waveguide quality}
\]

We should increase the waveguide quality \(h\)!

Fig. 11 Generation of a long \((b >> b_R)\), gradient-index, plasma-waveguide by a capillary Z-pinch
MHD for plasma waveguide: 2 temperatures ($T_e, T_i$) and 2 fluids ($n_e V_e, n_i V_i$)

\[
\begin{align*}
\frac{dn_e}{dt} + \text{div} (n_e V_e) &= 0 \quad (72) \\
\frac{dn_i}{dt} + \text{div} (n_i V_i) &= 0 \quad (73)
\end{align*}
\]

\[
\begin{align*}
m_e n_e \frac{dV_{e\alpha}}{dt} &= \frac{dP_{e\alpha}}{dx} - \frac{d\pi_{e\alpha\beta}}{dx} - e n_e (E_\alpha + (V_x B)_\alpha + R_\alpha) \\
m_i n_i \frac{dV_{i\alpha}}{dt} &= \frac{dP_{i\alpha}}{dx} - \frac{d\pi_{i\alpha\beta}}{dx} + \text{Ze} n_i (E_\alpha + (V_x B)_\alpha - R_\alpha) \quad (74, 75)
\end{align*}
\]

\[
\begin{align*}
\frac{3}{2} n_e \frac{dE}{dt} &= - P_e \text{div} V_e - \pi_{e\alpha\beta} \frac{dV_{e\alpha}}{dx} - \text{div} Q_e + W_e \quad (76) \\
\frac{3}{2} n_i \frac{dE}{dt} &= - P_i \text{div} V_i - \pi_{i\alpha\beta} \frac{dV_{i\alpha}}{dx} - \text{div} Q_i + W_i \quad (77)
\end{align*}
\]

\[
\begin{align*}
P_e &= n_e T_e, \quad P_i = n_i T_i \quad (81)
\end{align*}
\]

\[
\begin{align*}
\frac{d}{dt} &= \frac{d}{dt} + V_e (d/dr) \quad (82) \\
\frac{d}{dt} &= \frac{d}{dt} + V_i (d/dr) \quad (83)
\end{align*}
\]

For the adiabatic stage ($Q = 0$)

\[
\begin{align*}
P_e / P_{0e} &= \left( \frac{n_e}{n_{0e}} \right)^{5/3} \quad (79) \\
P_i / P_{0i} &= \left( \frac{n_i}{n_{0i}} \right)^{5/3} \quad (80)
\end{align*}
\]

For plasma waveguide, plasma viscosity was taken into account
(For details, see S.I. Braginskii, Transport processes in a plasma, Reviews of Plasma Physics, 1, AP. NY (1965)).
2-temperature \((T_e, T_i)\), 1-fluid \((\rho, V_r)\) 1-D MHD for the cylindrical Z-pinch plasma waveguide

\[
\begin{align*}
\frac{d\rho}{dt} &= -\rho \frac{1}{r} \frac{\partial}{\partial r} \left( rv_r \right) \\
\rho \frac{dv_r}{dt} &= -\frac{\partial P}{\partial r} - \frac{1}{c} j_z B_\phi - \frac{\partial}{\partial r} \Pi_{rr} - \frac{1}{r} \left( \Pi_{rr} - \Pi_{\phi \phi} \right) \\
\rho \frac{de_e}{dt} &= -\frac{P_e}{r} \frac{\partial}{\partial r} \left( rU_r \right) - q_e E^*_z + \frac{1}{r} \frac{\partial}{\partial r} \left( rU_{r \phi} \right) - Q_{\text{rad}} - C_{ei}
\end{align*}
\]

\[
\begin{align*}
\rho \frac{dE_i}{dt} &= -\frac{P_i}{r} \frac{\partial}{\partial r} \left( rv_r \right) - \frac{1}{r} \frac{\partial}{\partial r} \left( rU_{r \phi} \right) - \Pi_{rr} \frac{1}{r} \frac{\partial}{\partial r} \left( rU_r \right) - U_r \left( \Pi_{\phi \phi} - \Pi_{rr} \right)
\end{align*}
\]

\[
E^*_z = \frac{j_z}{\sigma_{\parallel}} - NB_\phi \frac{\partial T_e}{\partial r}
\]

\[
q_e = -k_e \frac{\partial T_e}{\partial r} + NT_e B_\phi j_z
\]

\[
E^*_z = E_z + \frac{U_r B_\phi}{c}
\]

\[
j_z = \frac{c}{2r\pi} \frac{\partial}{\partial r} \left( rB_\phi \right)
\]

\[
\frac{d}{dt} \frac{B}{r \rho} = \frac{c}{r \rho} \frac{\partial}{\partial r} E^*_z
\]

\[
\frac{d}{dr} \frac{\sim}{\sim} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r}
\]

For the cylindrical Z-pinch plasma waveguide, plasma viscosity was taken into account (For details, see S.I. Braginskii, Transport processes in a plasma, Reviews of Plasma Physics, 1. AP. NY (1965)).
Fig. 12 Results of the computer simulation of the $N_e$-profile of a plasma waveguide produced by the capillary z-pinch with the peak current $I_{\text{peak}} = 2.2$ kA and the half-cycle duration $T/2 = 170$ ns in a 4 mm-diameter 0.45 m-long capillary filled with H at the initial gas pressure 30 Torr and the different time moments $t$ (from [4])

$$h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \approx 0.01 \ (1\%)$$
**Ne\ (Z=10) plasma-based waveguide (A)**

**Fig. 13** The profiles of $N_e$ of a Ne (Z=10) plasma-based waveguide (U. Pecs). The highest quality h is at 17 ns.
The profiles of $N_e$ of a Neon (Z=10) plasma-based waveguide (U. Pecs). Neon plasma ($I=22kA$, $L=20cm$) with the electron density profiles computed at 2-15ns.

$$h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \sim 0.31 \ (31\%) \ \text{at} \ 17 \ \text{ns}$$
**Cl (Z=17) plasma-based waveguide**

**Fig. 15** The profiles of $N_e$ of a Cl (Z=17) plasma-based waveguide (U. Pécs)

$$h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \sim 0.31 \text{ (31\%)} \text{ at } 17 \text{ ns}$$
Argon (Z=18) plasma-based waveguide (A)

**Fig. 16** The profiles of $N_e$ obtained by the computer simulation of the Ar-plasma waveguide. The waveguide is produced by the capillary Z-pinch with the peak current $I_{\text{peak}} = 20$ kA and the half-cycle duration $T/2 = 120$ ns in a 3.2 mm-diameter 11 cm-long capillary filled with Ar at the initial gas pressure 0.25 Torr and the different time moments $t$ (from [4])

$$h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \sim 0.2 \ (20\%)$$
Argon (Z=18) plasma-based waveguide (B)

**Fig. 17** The profiles of $N_e$ of an Argon plasma (Z=18) based waveguide (U. Pecs).
The waveguide exists during 13-31 ns.

\[ h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \sim 0.45 \text{ (45%)} \]
**Xe (Z=54) plasma-based waveguide**

![Graph](image)

**Fig. 18** The profiles of $N_e$ of a Xe (Z=54) based plasma waveguide (U. Pécs)

$$h = \frac{N_e(r_w) - N_e(0)}{N_e(r_w)} \approx 0.47 \, (47\%) \text{ at } 13 \, \text{ns}$$

Thus we can achieve a high quality $h(Z)$ of the plasma waveguide! Notice, for the acceleration process, phase matching should be provided by the following effect

$$\Delta \varphi = \frac{2\pi}{\lambda} \int_{n_1}^{n_2} (n-1) dl = \frac{2\pi}{\lambda} \int_{n_1}^{n_2} \left(1 - \frac{n_e}{n_0}\right)^{1/2} - 1 \, dl$$

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
Set-up for plasma waveguides for electron acceleration based on Z-pinch at UP (A)

Fig. 19 Set-up for R&D of capillary Z-pinch plasma-based waveguides (A) (U. of Pécs).
Set-up for R&D of plasma waveguides for electron acceleration based on capillary Z-pinch at University of Pécs (B)

The experiments are in progress.

Fig. 20 Set-up for R&D of plasma waveguides for electron acceleration based on the capillary z-pinch at University of Pécs (B).
Understanding waves in Z-pinches requires theory, computations and experiments.

For an example:
Artificial adjustments of parameters in the models

Why can the 25-year theoretical and experimental experience of the University of Pécs in capillary-based discharges, plasmas, waveguides and x-ray waveguide optics be useful for R&D and applications of the plasma-waveguide based accelerators?

Fig. 21 Understanding the waveguiding by Z-pinch requires theory, computations and experiments.
Problems as home assignments (A)

1. Compare different kinds of plasmas. Are they kinetic or collective?
2. Explain the usefulness of waves in Z-pinch capillary plasma for applications.
3. Describe the basic physical processes related to waves in Z-pinch capillary plasma.
4. Explain the wave-particle interactions in Z-pinch capillary plasmas.
5. Explain the kinetic effect associated with Landau’s damping or growth in Z-pinch capillary plasmas.
6. Explain schematically linear and non-linear processes in Z-pinch plasmas using wave scattering as an example.
7. Explain electro-acoustic waves in Z-pinch capillary plasmas by using the dispersion relation and diagram.
8. Consider the transverse electromagnetic waves in Z-pinch capillary plasmas.
9. Qualitatively describe propagation of TEM waves in overdense Z-pinch capillary plasmas.
10. Explain the connection of the refractive index of Z-pinch capillary plasma with phase velocity and group velocity of TEM waves in Z-pinch capillary plasma.
11. Describe the collisional absorption of a TEM wave in Z-pinch capillary plasma.
13. Describe in details linear and nonlinear processes in a Z-pinch capillary plasma using scattering as an example.
15. Consider an example of the stimulated Raman backscattering at $n_e = n_e/4$.
16. Way can very hard x-rays be generated in a Z-pinch capillary plasma by intense laser radiation?
17. How are R&D of plasma waveguides for electron acceleration in the Hungarian-ELI project related to ~25-years theoretical and experimental results obtained in the field of capillary waveguides, capillary discharges and capillary z-pinches at University of Pecs (UP)?
18. Describe how light confinement is limited by diffraction.
19. Why is a plasma waveguide by Z-pinch, capillary Z-pinch or capillary discharge very suitable for electron acceleration?
20. Schematically describe a plasma-based waveguide by the MHD model.
22. Give an example of 2-temperature ($T_e$ and $T_i$), 2-fluid ($N_e, V_e$ and $N_i, V_i$) MHD-model for Z-pinch waveguide.
23. Give and example of 2-temperature ($T_e, T_i$), 1-fluid ($r, V_r$) 1-D MHD-model for Z-pinch plasma-based waveguide.
24. Why does understanding waves in Z-pinch waveguide require theory, computations and experiments?
24. Explain the advantages and disadvantages of Hydrogen (Z=1), Cl (Z=17), Argon (Z=18) and Xenon (Xe Z=54) plasma-based waveguides computed by the MHD model of the U. of Pécs.
25. Describe the experimental set-up developed at the U. of Pécs for R&D of plasma waveguides for electron acceleration based on Z-pinch.
26. Why does understanding capillary Z-pinch waveguides require theory, computations and experiments?
References

1. Abonyi Iván, A negyedik halmazállapot. Bevezető a plazmafizikába, Gondolat kiadó, Budapest (1971)

For additional information see: