Lecture 6

X-ray sources based on Crookes tubes and Crookes-like capillary plasma tubes

Outline


• Electron binding energies [eV] for atoms in their natural forms (relevant to K, L,M,… lines). A lot of x-ray emission lines from multiple-charged ions (MCIs).

• Crookes tube vs capillary Crookes-like discharge: Generation of high-intensity x-rays.

• Ionization energies $E_{\text{ionization}}$ [eV] of atoms and multiple charged ions (MCI). X-ray spectra using balance equations for the ion stages and energy levels of uniform plasma.

• Three plasma-physics models of Crooke plasma and capillary Crookes-like plasma. Microscopic model by using the Dirac delta distribution function of a point-like particle. Microscopic model like a kinetic (Klimontovich) model of Crookes and Crookes-like plasmas.

• Kinetic model by using Vlasov equation Kinetic model by using Vlasov and Maxwell equations for Crookes or Crookes-like plasma.
Outline ctd.

- MHD for capillary Crookes-like plasma with 2 temperatures \((T_e, T_i)\) and 2 fluids \((n_e, V_e, n_i, V_i)\). MHD for Crookes or Crookes-like plasma. Continuity equation for the conservation of mass or particles in MHD of Crookes or Crookes-like plasma. The force equation of conservation of momentum in MHD of Crookes or Crookes-like plasma. The equation of conservation of energy in MHD of Crookes or Crookes-like plasma. Balance equations for the energy levels and ion stages for Crookes or Crookes-like plasma.


- The use of Van Cittert-Zernike theorem for x-rays generated by Crookes-like plasma.

- Understanding capillary Crookes-like plasma sources requires theory, computations and experiments.

- Problems as home assignments

- References
Plasma as $4^{th}$ state of matter

$$T(K) \rightarrow T(eV) = k_B T$$

$11600 \text{ K} \sim 1 \text{ eV}$

$T_c \sim 0.25 \text{ eV}$

Degree of ionization

$x = N_{ch}/N_0$

$x = 0$ – gas

$0 < x < 1$ – non-perfect plasma

$x = 1$ – perfect plasma (plasma)

**Fig. 1** Plasma as $4^{th}$ state of matter

1. For the local thermo-dynamical equilibrium (LTE) with the Boltzmann statistics $N_m = N_0 A e^{-\frac{E_m}{k_B T}}$

Saha equation

$$\frac{x^2}{1-x^2} = 2A \left( \frac{2\pi m}{h^2} \right)^{3/2} [k_B T/P] e^{-\frac{U_i}{k_B T}}$$

$$\Rightarrow T_c$$

2. For a general case $x$ by:

balance equation of ionization and recombination

$$\Rightarrow T_c$$

**Fig. 2** Degree of ionization $x$ of hydrogen plasma vs temperature $T$ (Figure 2 from Abonyi Iván, A negyedik halmazállapot. Bevezető a plazmafizikába, Gondolat kiadó, Budapest (1971))
Kinds of plasmas

Basic plasma parameters:

- $T_e$ [K]
- $k_B T_e$ [eV]
- $n_e$ [cm$^{-3}$]
- $\omega_p$ [rad/sec]

\[
\omega_p = \left(\frac{4\pi n_e e^2}{m}\right)^{1/2}
\]

\[
\lambda_D = \left(\frac{k_B T}{4\pi n_e e^2/m}\right)^{1/2}
\]

\[
N_D = \left(\frac{4\pi}{3}\right)n_e \lambda_D^3
\]

Fig. 3 Plasmas of charges. Kinds of plasmas (Figure from Abonyi Iván. A negyedik halmazállapot. Bevezető a plazmazfizikába. Gondolat kiadó, Budapest (1971)).

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
**Crookes X-tube in the frame of Townsend model**

Let us consider Crookes tubes in the context of transition from incoherent to coherent x-ray sources.

**Fig. 4** Crookes x-ray tube (electron beam = cathode beam)

**Fig. 5** X-rays from a Crookes tube (glow discharge)

---

**Quantitative description of Crookes X-ray tube in Townsend model of ionization**

1. **Townsend model** of glow plasma.
   - Ionization and plasma current:
   
   \[ \frac{I}{I_0} = e^{\alpha d} \quad [1 - \gamma (e^{\alpha d} - 1)] \]

   \[ \alpha = \frac{1}{\langle \lambda \rangle} \exp \left[ -\frac{U_i}{eE \lambda} \right] \]

   - \( U_i \) – ionization potential
   - \( e \) – electron charge
   - \( E \) – electric field
   - \( \langle \lambda \rangle \) – mean free path

   - \( \alpha \) - 1-st Townsend avalanche ionization coefficient by anion (in volume)
   - \( \gamma \) - Townsend 2-nd ionization coefficient by ion (on the cathode surface)
   - \( d \) - distance between electrodes, \( P \) - gas pressure
   - \( I_0 \) – initial current

---

**Paschen Curve:**

\[ U_b = \frac{A d}{\ln(P d) + B} \]

---

**TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project**
K, L, M... lines from a Crookes-tube

1st regime: The mean free path \( \langle \lambda \rangle \) in a Crookes tube is given by \( \langle \lambda \rangle > d \), where \( d \) is the distance between the electrodes. Then, Crookes tube \( \rightarrow \) vacuum tube. if \( \frac{mv^2}{2} \sim e\Delta \phi > E_b \). In such a case, the Crookes tube generates the K, L, M photons with energies \( h\omega_{K,L,M} \).

2nd regime: The mean free path \( \langle \lambda \rangle \) in a Crookes tube is given by \( \langle \lambda \rangle < d \). Then, vacuum tube \( \rightarrow \) Crookes tube if \( \frac{p^2}{2m} \sim e\Delta \phi (\lambda/d) > E_b \). In such a case, the Crookes tube generates the K, L, M photons with energies \( h\omega_{K,L,M} \).

The mean free path in a Crookes tube
\[ \lambda \sim \frac{1}{p} \sim n^{-1/3} \]

Rutherford model gives
\[ \lambda = 9k_B T^2 / \pi n e^4 \]
**K, L... lines and bremsstrahlung in 1st regime**

$E_b$ is the electron binding energy

Shells: K, L, M, N, ...

$n=1$ (K
$n=2$ (L
$n=3$ (M
$n=4$ (N
$(n=\text{inf.})$ (Continuum)

 Photon $K\alpha$

Fig. 7 Ionization of an anode atom by the avalanche electron and subsequent radiation of a $K\alpha$-line photon

1st regime (characteristic lines by anode atoms):

Since $\langle \lambda \rangle > d$ and $\langle \lambda \rangle_{\text{Rutherford}} = 9k_B^2T^2/\pi ne^4$, then

$$T > (d \pi ne^4/9k_B^2)^{1/2}$$

(1)

and

$$e\Delta\phi > E_b$$

(2)

1st regime (bremsstrahlung by the anode atoms):

$E(r, t) \sim e a_T(t - r/c) / r$
**K, L, ... lines and bremsstrahlung in 2\textsuperscript{nd} regime**

$E_b$ is the electron binding energy

Shells: K, L, M, N, ... (n=inf.)

N (n=4)

M (n=3)

L (n=2)

L\textsubscript{\alpha}, L\textsubscript{\beta}, K\textsubscript{\gamma}

Photon K\textsubscript{\alpha}

Bremsstrahlung + recombination

Fig. 8 Ionization of a gas atom by the avalanche electron and subsequent radiation of a K\textsubscript{\alpha}-line photon

2\textsuperscript{nd} regime (characteristic lines by gas atoms):

Since $<\lambda> <d$ and $<\lambda>_{\text{Rutherford}} = 9k_B^2 T^2 / \pi ne^4$,

then $T > (d \pi ne^4 / 9k_B^2)^{1/2}$ (3)

and $e\Delta\phi (\lambda/d) > E_b$ (4)

1\textsuperscript{st} regime (bremsstrahlung by the gas atoms):

Bremsstrahlung by an anode atom

Bremsstrahlung radiation intensity ([eV/cm\textsuperscript{3}]):

$I_B \sim 1.5 \times 10^{-27} N_e N_i Z^2 (T_e)^{1/2}$

Recombination radiation intensity ([eV/cm\textsuperscript{3}]):

$I_R \sim 5 \times 10^{-27} N_e N_i Z^4 (T_e)^{-1/2}$

Electron binding energies [eV] for atoms in their natural forms (relevant to \( K, L, M, \ldots \) lines)

**Tab. 1** Electron binding energies, in electron volts, for the elements in their natural forms.

<table>
<thead>
<tr>
<th>Element</th>
<th>K 1s</th>
<th>L 1 2s</th>
<th>L 2 2p1/2</th>
<th>L 3 2p3/2</th>
<th>M 1 3s</th>
<th>M 2 3p1/2</th>
<th>M 3 3p3/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>13.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>He</td>
<td>24.6*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Li</td>
<td>54.7*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Be</td>
<td>111.5*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>188*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>284.2*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>409.9*</td>
<td>37.3*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>543.3*</td>
<td>41.6*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>F</td>
<td>696.7*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ne</td>
<td>870.2*</td>
<td>48.5*</td>
<td>21.7*</td>
<td>21.6*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Na</td>
<td>1070.8†</td>
<td>63.5†</td>
<td>30.65</td>
<td>30.81</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mg</td>
<td>1303.0†</td>
<td>88.7</td>
<td>49.78</td>
<td>49.50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Al</td>
<td>1559.6</td>
<td>117.8</td>
<td>72.95</td>
<td>72.55</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Si</td>
<td>1839</td>
<td>149.7*</td>
<td>99.82</td>
<td>99.42</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P</td>
<td>2145.5</td>
<td>189*</td>
<td>136*</td>
<td>135*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td>2472</td>
<td>230.9</td>
<td>163.6*</td>
<td>162.5*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cl</td>
<td>2822.4</td>
<td>270*</td>
<td>202*</td>
<td>200*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ar</td>
<td>3205.9</td>
<td>376.3*</td>
<td>250.6†</td>
<td>248.4*</td>
<td>29.3*</td>
<td>15.9*</td>
<td>15.7*</td>
</tr>
<tr>
<td>K</td>
<td>3608.4*</td>
<td>378.6*</td>
<td>297.3*</td>
<td>294.6*</td>
<td>34.8*</td>
<td>18.3*</td>
<td>18.3*</td>
</tr>
<tr>
<td>Ca</td>
<td>4038.5*</td>
<td>438.4†</td>
<td>349.7†</td>
<td>346.2†</td>
<td>44.3†</td>
<td>25.4†</td>
<td>25.4†</td>
</tr>
<tr>
<td>Sc</td>
<td>4492</td>
<td>498.0*</td>
<td>403.6*</td>
<td>398.7*</td>
<td>51.1*</td>
<td>28.3*</td>
<td>28.3*</td>
</tr>
<tr>
<td>Ti</td>
<td>4966</td>
<td>560.9†</td>
<td>460.2†</td>
<td>453.8†</td>
<td>58.7*</td>
<td>32.6†</td>
<td>32.6†</td>
</tr>
<tr>
<td>V</td>
<td>5465</td>
<td>626.7†</td>
<td>519.8†</td>
<td>512.1†</td>
<td>66.3†</td>
<td>37.2†</td>
<td>37.2†</td>
</tr>
<tr>
<td>Cr</td>
<td>5989</td>
<td>696.0†</td>
<td>583.8†</td>
<td>574.1†</td>
<td>71.4†</td>
<td>42.2†</td>
<td>42.2†</td>
</tr>
<tr>
<td>Mn</td>
<td>6539</td>
<td>769.1†</td>
<td>649.9†</td>
<td>638.7†</td>
<td>82.3†</td>
<td>47.2†</td>
<td>47.2†</td>
</tr>
<tr>
<td>Fe</td>
<td>7112</td>
<td>844.6†</td>
<td>719.9†</td>
<td>706.8†</td>
<td>91.3†</td>
<td>52.7†</td>
<td>52.7†</td>
</tr>
<tr>
<td>Co</td>
<td>7709</td>
<td>925.1†</td>
<td>793.2†</td>
<td>778.1†</td>
<td>101.0†</td>
<td>58.9†</td>
<td>59.9†</td>
</tr>
<tr>
<td>Ni</td>
<td>8333</td>
<td>1008.6†</td>
<td>870.0†</td>
<td>852.7†</td>
<td>110.8†</td>
<td>68.0†</td>
<td>66.2†</td>
</tr>
<tr>
<td>Cu</td>
<td>8979</td>
<td>1096.7†</td>
<td>952.3†</td>
<td>932.7†</td>
<td>122.5†</td>
<td>72.3†</td>
<td>75.1†</td>
</tr>
</tbody>
</table>

The binding energies can be used for quantitative modeling of X-tubes with the Cu anode and Ar gas.

(Fig. 9 and Table 1 from http://xdb.lbl.gov)

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
A lot of x-ray emission lines from multiple-charged ions (MCIs)

Fig. 10 Multiple-charged ions (MCIs) radiating a lot of x-ray emission lines.
**Crookes tube vs Crookes-like capillary discharge: Generation of high-intensity x-rays**

The intensity of x-rays from the Crookes (\(P_{\text{air}} \approx 7 \times 10^{-4} - 4 \times 10^{-5}\) Torr, \(U > 5\) kV) glow-discharge plasma is weak.

The intensity of x-rays from Crookes-like (\(P_{\text{gas}} \approx 1\) Torr, \(U > 5\) kV) powerful capillary discharge is strong.

**Fig. 11** X-rays from Crookes-like (glow-like) powerful capillary discharge.
<table>
<thead>
<tr>
<th>Ionization energies [eV] of atoms and multiple charged ions (MCIs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen H</td>
</tr>
<tr>
<td>Helium He</td>
</tr>
<tr>
<td>Lithium Li</td>
</tr>
<tr>
<td>Beryllium Be</td>
</tr>
<tr>
<td>Boron B</td>
</tr>
<tr>
<td>Carbon C</td>
</tr>
<tr>
<td>Nitrogen N</td>
</tr>
<tr>
<td>Oxygen O</td>
</tr>
<tr>
<td>Fluorine F</td>
</tr>
<tr>
<td>Neon Ne</td>
</tr>
<tr>
<td>Sodium Na</td>
</tr>
<tr>
<td>Magnesium Mg</td>
</tr>
<tr>
<td>Aluminum Al</td>
</tr>
<tr>
<td>Silicon Si</td>
</tr>
<tr>
<td>Phosphorus P</td>
</tr>
<tr>
<td>Sulfur S</td>
</tr>
<tr>
<td>Chlorine Cl</td>
</tr>
<tr>
<td>Argon Ar</td>
</tr>
<tr>
<td>Potassium K</td>
</tr>
<tr>
<td>Calcium Ca</td>
</tr>
<tr>
<td>Scandium Sc</td>
</tr>
<tr>
<td>Titanium Ti</td>
</tr>
<tr>
<td>Vanadium V</td>
</tr>
<tr>
<td>Chromium Cr</td>
</tr>
<tr>
<td>Manganese Mn</td>
</tr>
<tr>
<td>Iron Fe</td>
</tr>
<tr>
<td>Cobalt CO</td>
</tr>
<tr>
<td>Nickel Ni</td>
</tr>
<tr>
<td>Copper Cu</td>
</tr>
<tr>
<td>Zinc Zn</td>
</tr>
<tr>
<td>Gallium Ga</td>
</tr>
<tr>
<td>Germanium Ge</td>
</tr>
<tr>
<td>Arsenic As</td>
</tr>
<tr>
<td>Selenium Se</td>
</tr>
<tr>
<td>Bromine Br</td>
</tr>
<tr>
<td>Krypton Kr</td>
</tr>
<tr>
<td>Rubidium Rb</td>
</tr>
<tr>
<td>Strontium Sr</td>
</tr>
<tr>
<td>Yttrium Y</td>
</tr>
<tr>
<td>Zirconium Zr</td>
</tr>
<tr>
<td>Niobium Nb</td>
</tr>
<tr>
<td>Molybdenum Mo</td>
</tr>
<tr>
<td>Technetium Tc</td>
</tr>
<tr>
<td>Ruthenium Ru</td>
</tr>
<tr>
<td>Rhodium Rh</td>
</tr>
<tr>
<td>Palladium Pd</td>
</tr>
<tr>
<td>Silver Ag</td>
</tr>
<tr>
<td>Cadmium Cd</td>
</tr>
<tr>
<td>Indium In</td>
</tr>
<tr>
<td>Tin Sn</td>
</tr>
<tr>
<td>Antimony Sb</td>
</tr>
<tr>
<td>Tellurium Te</td>
</tr>
<tr>
<td>Iodine I</td>
</tr>
<tr>
<td>Xenon Xe</td>
</tr>
</tbody>
</table>

**Table 2** Ionization energies [eV] of atoms and multiple charged ions (from http://dept.astro.lsa.umich.edu/~cowley/ionen.htm)

Ionization "bottlenecks" limit the degree of ionization of atoms and MCIs in a Crookes plasma if

\[ \frac{p^2}{2m} \sim e\Delta \varphi \left( \frac{\lambda}{d} \right) < E_{\text{ionization}} \] (5)
X-ray spectra by using balance equations for ion stages and energy levels of uniform plasma

Balance equations for the ion stages assuming the known values of $T_e(t), T_i(t), n_e(t)$ and $n_i(T)$

$$
\frac{dn_{ion}^Z}{dt} = n_e \cdot \left\{ n_{ion}^{Z-1} I^{Z-1} + n_{ion}^{Z+1} \left( R_{rr}^{Z+1} + R_{dr}^{Z+1} + R_{cr}^{Z+1} \right) - n_{ion}^Z I^Z - n_{ion}^Z \left( R_{rr}^Z + R_{dr}^Z + R_{cr}^Z \right) \right\} \quad (6)
$$

Balance equations for the energy levels assuming the known values of $T_e(t), T_i(t), n_e(t)$ and $n_i(T)$

$$
\frac{d(n_{ion}^Z)_{ji}}{dt} = -(n_{ion}^Z)_{ji} \left[ \sum_{i<j} (A_{ji} + P_{dji}^e) + \sum_{i>j} P_{jii}^e \right] + \sum_{i>j} (n_{ion}^Z)_{ij} (A_{ji} + P_{dji}^d) + \sum_{i<j} (n_{ion}^Z)_{ij} P_{jii}^e \quad (7)
$$

$$
A_{ji} \rightarrow E(\tau)A_{ji}, \quad (8)
$$

$$
E(\tau) = \frac{1}{2} \int_{-X}^{X} \left[ \int_{0}^{1} \exp\left[ -(x - y\beta\tau)^2 \right] \exp\left[ \frac{(-1)^2}{y} \int_{0}^{2\tau} \exp\left[ -(x - y\beta)z^2 \right] dz \right] dy \right] dx \quad (9)
$$

I-ionization rate coefficients, R- recombination rate coefficients, A -Einstein coefficient, E(τ)- photon escape factor, P-electron excitation /de-excitation coefficients

How can we find $T_e(r,t), T_i(r,t), n_e(r,t)$ and $n_i(r,T)$?
Three plasma-physics models of a Crooke plasma and a Crookes-like capillary plasma

1st: Microscopic model
for all particles

\[ r_i(t), \]
\[ v_i(t) = \frac{dr_i(t)}{dt}, \]
\[ a_i(t) = \frac{dv_i(t)}{dt} = \frac{q_i}{m_i} [E + v_i \times B] \]

The 1st model is based on the use of the Lorentz force \( ma = q(E + v \times B) \).

2nd: Kinetic model
for the spatio-temporal evolution of the distribution function

\[ \bar{f}(v, r, t) + \text{Maxwell equations} \]

\[ \bar{f}(v, r, t) = \frac{N}{(2\pi)^{3/2}} e^{-mv^2/2k_BT} \]

For the time-independent, homogeneity case we have the Maxwell distribution

The 2nd model is based on the use of a concept of the distribution function determined by

\[ \int \int f(v, r, t) dV dR = N \quad (10) \]

where \( N \) is the particle number, which determines the particle density as

\[ \int f(v) dV = n \quad (11) \]

For the non-collective motion we have

\[ E_K \gg E_p, \quad (1/2)k_BT \sim e^2/\langle \lambda \rangle, \quad \langle \lambda \rangle \sim n^{-1/3} \]

Collective motion

\[ E_K \sim E_p, \quad (1/2)k_BT \sim e^2/\langle \lambda \rangle, \quad \langle \lambda \rangle \sim n^{-1/3} \]
Microscopic model by using the Dirac delta distribution function of a point-like particle

In Electrodynamics, an electron in a point \( \mathbf{r} \) is described by the point-like charge density

\[
\rho_i = q_i n_i(\mathbf{r} - \mathbf{r}_i) = q_i \delta(\mathbf{r} - \mathbf{r}_i)
\]

which yields

\[
f(\mathbf{v}, \mathbf{r}, t) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i(t))\delta(\mathbf{v} - \mathbf{v}_i(t))
\]

Then the time evolution of the distribution function is determined as

\[
\frac{\partial}{\partial t} f(\mathbf{v}, \mathbf{r}, t) = \sum_i \left[ \frac{\partial}{\partial t} \delta(\mathbf{r} - \mathbf{r}_i(t)) \right] \delta(\mathbf{v} - \mathbf{v}_i(t)) \left[ \frac{\partial}{\partial \mathbf{v}} \delta(\mathbf{v} - \mathbf{v}_i(t)) \right] \delta(\mathbf{r} - \mathbf{r}_i(t)) = (15)
\]

\[
= \sum_i \left[ -\frac{\partial \mathbf{r}_i}{\partial t} \frac{\partial}{\partial \mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \right] \delta(\mathbf{v} - \mathbf{v}_i(t)) \left[ -\frac{\partial \mathbf{v}_i}{\partial t} \frac{\partial}{\partial \mathbf{v}} \delta(\mathbf{v} - \mathbf{v}_i(t)) \right] \delta(\mathbf{r} - \mathbf{r}_i(t)) = (16)
\]

\[
= \sum_i \left[ -\mathbf{v}_i \frac{\partial}{\partial \mathbf{r}} \delta(\mathbf{r} - \mathbf{r}_i(t)) \right] \delta(\mathbf{v} - \mathbf{v}_i(t)) \left[ -\mathbf{a}_i \frac{\partial}{\partial \mathbf{v}} \delta(\mathbf{v} - \mathbf{v}_i(t)) \right] \delta(\mathbf{r} - \mathbf{r}_i(t))
\]

Then we get the microscopic description of plasma by the Klimontovich equation

\[
\frac{\partial}{\partial t} f + \mathbf{v} \frac{\partial}{\partial \mathbf{r}} f + \mathbf{a} \frac{\partial}{\partial \mathbf{v}} f = \]

\[
= \frac{\partial}{\partial t} f + \mathbf{v} \nabla f + \frac{q_i}{m_i} [\mathbf{E} + \mathbf{v} \times \mathbf{B}] \nabla \cdot f = 0
\]
Microscopic model as a kinetic (Klimontovich) model of Crookes and Crookes-like plasmas

It should be stressed that the microscopic description of plasmas and the Klimontovich equation is applicable also to other kinds of plasmas (not only to a Crookes plasma or a Crookes-like capillary plasma).

The relation between the microscopic description of plasma and the Klimontovich equation is demonstrated by the following equations

\[
f(v, r, t) = \sum^n_i \delta(r - r_i(t)) \delta(v - v_i(t))
\]

\[
\frac{\partial}{\partial t} f + v \nabla f + \frac{q_i}{m_i} [E + v_i \times B] \frac{\partial}{\partial v} f = 0
\]

\[
v_i(t) = \frac{d r_i(t)}{dt}
\]

\[
a_i(t) = \frac{d v_i(t)}{dt} = \frac{q_i}{m_i} [E + v_i \times B]
\]
Kinetic model using Vlasov equation

Let us assume that the distribution function of the plasma can be presented as a superposition of the slowly varying part and a rapidly fluctuating part:

\[ f(v, r, t) = \bar{f}(v, r, t) + \tilde{f}(v, r, t) \]  

(25)

Similarly, we suppose that fields \( E \) and \( B \) can be presented as the sums of the slowly and rapidly varying components:

\[ E(v, r, t) = \bar{E}(v, r, t) + \tilde{E}(v, r, t) \]  

(26)

\[ B(v, r, t) = \bar{B}(v, r, t) + \tilde{B}(v, r, t) \]  

(27)

Substituting Eqs. (25)-(27) into the Klimontovich equation (20) and the subsequent spatial averaging yields the collisional Vlasov equation

\[
\frac{\partial}{\partial t} \bar{f} + v \nabla \bar{f} + \frac{q_i}{m_i} [E + v \times B] \nabla_v \bar{f} = - \frac{q_i}{m_i} [E + v \times B] \nabla_v \bar{f}
\]  

(28)

The use Eq. (28) for electron and ions yields the collision-less Vlasov equation

\[
\frac{\partial}{\partial t} \bar{f} + v \nabla \bar{f} + \frac{q_i}{m_i} [E + v \times B] \nabla_v \bar{f} = 0
\]  

(29)
Kinetic model using Vlasov and Maxwell equations for Crookes or Crookes-like plasma

Thus a kinetic model, which uses Vlasov equation and Maxwell equations for a Crookes plasma or a Crookes-like capillary plasma, is given by the equation

\[ \frac{\partial}{\partial t} \tilde{f} + \mathbf{v} \nabla \tilde{f} + \frac{q_i}{m_i} \left[ \mathbf{E} + \mathbf{v} \times \mathbf{B} \right] \nabla_v \tilde{f} = 0 \]  

(30)

coupled with Maxwell equations. In the equations, the sums are over all kinds of charged particles.

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]  

(31)

\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \sum_j q_j \int \mathbf{v} \tilde{f}_j (\mathbf{v}, \mathbf{r}, t) d\mathbf{v} \]  

(32)

\[ \nabla \mathbf{D} = \sum_j q_j \int \tilde{f}_j (\mathbf{v}, \mathbf{r}, t) d\mathbf{v} \]  

(33)

\[ \nabla \mathbf{B} = 0 \]  

(34)

\[ \mathbf{D} = \varepsilon_0 \mathbf{E} \]  

(35)

\[ \mathbf{B} = \mu_0 \mathbf{H} \]  

(36)

It should be stressed that the microscopic kinetic model of plasma is applicable also to other kinds of plasmas (not only to a Crookes plasma or a Crookes-like capillary plasma).
MHD for a Crookes-like capillary plasma with
2 temperatures \((T_e, T_i)\) and 2 fluids \((n_e, V_e, n_i, V_i)\)

\[
\begin{align*}
\frac{dn_e}{dt} + \text{div} (n_e V_e) &= 0 \tag{37} \\
\frac{dn_i}{dt} + \text{div} (n_i V_i) &= 0 \tag{38} \\
m_e n_e \frac{dV_e}{dt} &= \frac{dP_e}{dx} - en_e (E_e + (V_e \times B)_e) + R_e \tag{39} \\
m_i n_i \frac{dV_i}{dt} &= \frac{dP_i}{dx} + Zen_i (E_i + (V_i \times B)_i) - R_i \tag{40} \\
(3/2)n_e \frac{dT_e}{dt} &= - P_e \text{div} V_e - \text{div} Q_e + W_e \tag{41} \\
(3/2)n_i \frac{dT_i}{dt} &= - P_i \text{div} V_i - \text{div} Q_i + W_i \tag{42} \\
P_e &= n_e T_e, \quad P_i = n_i T_i \tag{43} \\
\frac{d_e}{dt} &= \frac{d}{dt} + V_e \left( \frac{d}{dr} \right) \tag{44} \\
\frac{d_i}{dt} &= \frac{d}{dt} + V_i \left( \frac{d}{dr} \right) \tag{45} \\
\end{align*}
\]

For Crookes plasma or Crookes-like capillary plasma, plasma viscosity is neglected.
(For details, see S.I. Braginskii, Transport processes in a plasma, Reviews of Plasma Physics, 1, AP, NY (1965)).
**MHD for Crookes or Crookes-like plasma**

It should be stressed that MHD of plasmas is applicable also to other kinds of plasmas (not only to a Crookes plasma or a Crookes-like capillary plasma). The equations of magnetohydrodynamics (MHD) of a Crookes plasma or a Crookes-like capillary plasma are derived by using the equalities

\[
\int 1 \cdot \left[ \frac{\partial}{\partial t} \tilde{f} + \mathbf{V} \nabla \tilde{f} + \frac{q_i}{m_i} \left[ \mathbf{E} + \mathbf{V} \times \mathbf{B} \right] \nabla \mathbf{v} \cdot \tilde{f} \right] d\mathbf{V} = 0
\]

(47)

\[
\int \mathbf{V} \cdot \left[ \frac{\partial}{\partial t} \tilde{f} + \mathbf{V} \nabla \tilde{f} + \frac{q_i}{m_i} \left[ \mathbf{E} + \mathbf{V} \times \mathbf{B} \right] \nabla \mathbf{v} \cdot \tilde{f} \right] d\mathbf{V} = 0
\]

(48)

\[
\int \frac{m\mathbf{V}^2}{2} \cdot \left[ \frac{\partial}{\partial t} \tilde{f} + \mathbf{V} \nabla \tilde{f} + \frac{q_i}{m_i} \left[ \mathbf{E} + \mathbf{V} \times \mathbf{B} \right] \nabla \mathbf{v} \cdot \tilde{f} \right] d\mathbf{V} = 0
\]

(49)

obtained from the Vlasov equation (29) with the notation \( \mathbf{v} \equiv \mathbf{V} \). The equation (47) yields the continuity equations for electrons (37) and ions (38). The equation (48) yields the conservation momentum equations for electrons (39) and ions (40). The conservation energy equations for electrons (41) and (42) are obtained from Eq. (49).
Continuity equation for the conservation of mass or particles in MHD of Crookes or Crookes-like plasma

The calculation of Eq. (47) yields

\[
\left( \int \left[ \frac{\partial}{\partial t} \vec{f} \right] d\mathbf{V} \right) + \left[ \int [\mathbf{V} \nabla \vec{f}] d\mathbf{V} \right] + \left\{ \int \left[ \frac{q_i}{m_i} [\mathbf{E} + \mathbf{V} \times \mathbf{B}] \nabla_{\mathbf{v}} \vec{f} \right] \right\} = 0
\] (50)

\[
\left( \frac{\partial n}{\partial t} \right) + [\text{div}(n \cdot \mathbf{V})] + \{0\} = 0
\] (51)

\[
\frac{\partial n}{\partial t} + \text{div}(n \cdot \mathbf{V}) = 0
\] (52)

The continuity equation (52) for the electron and ion fluids yields respectively

\[
dn_e/\text{dt} + \text{div} (n_e \mathbf{V}_e) = 0 \\
dn_i/\text{dt} + \text{div} (n_i \mathbf{V}_i) = 0
\] (53) (54)
The force equation for conservation of momentum in MHD of a Crookes or Crookes-like plasma

For a Crookes-like, viscosity-less capillary plasma, the calculation of Eq. (48) yields

\[ \left( \int mV \left[ \frac{\partial}{\partial t} \vec{f} \right] dV \right) + \left[ \int mV \left[ \nabla \vec{f} \right] dV \right] + \left\{ \int mV \left[ \frac{q_i}{m_i} \left[ E + V \times B \right] \nabla \vec{f} \right] \right\} = 0 \]  \hspace{1cm} (55)

\[ mn \frac{dV_\alpha}{dt} = dP_\alpha / dx_\alpha - en (E_\alpha + (V \times B)_\alpha + R_\alpha) \]  \hspace{1cm} (56)

The force equation of conservation of momentum (55) for the electron and ion fluids yields respectively

\[ m_e n_e \frac{d_e V_{e\alpha}}{dt} = dP_{e\alpha} / dx_\alpha - en_e (E_\alpha + (V_e \times B)_\alpha + R_\alpha) \]  \hspace{1cm} (57)

\[ m_i n_i \frac{d_i V_{i\alpha}}{dt} = dP_{i\alpha} / dx_\alpha - Zn_i (E_\alpha + (V_i \times B)_\alpha - R_\alpha) \]  \hspace{1cm} (58)
The equation of conservation of energy in MHD of Crookes or Crookes-like plasma

For a Crookes-like, viscosity-less plasma, the calculation of Eq. (49) yields

\[
\frac{m}{2} \left[ \frac{\partial}{\partial t} \mathbf{f} \right] d\mathbf{V} + \left( \int \frac{m}{2} \mathbf{V} \nabla \mathbf{f} d\mathbf{V} \right) + \left\{ \int \frac{m}{2} \mathbf{V} \left[ \frac{q_i}{m_i} \left[ \mathbf{E} + \mathbf{V} \times \mathbf{B} \right] \nabla \mathbf{f} \right] d\mathbf{V} \right\} = 0
\]

(59)

\[(3/2)n \, dT/dt = - P \, \text{div} \, \mathbf{V} - \text{div} \, \mathbf{Q} + W \]

(60)

The equation of conservation of energy (60) for the electron and ion fluids yields respectively

\[
(3/2)n_e \, dT_e/dt = - P_e \, \text{div} \, \mathbf{V}_e - \text{div} \, \mathbf{Q}_e + W_e
\]

(61)

\[
(3/2)n_i \, dT_i/dt = - P_i \, \text{div} \, \mathbf{V}_i - \text{div} \, \mathbf{Q}_i + W_i
\]

(62)

It should be stressed again that the MHD equations (37)-(46) of plasmas are applicable also to other kinds of plasmas (not only to Crookes or Crookes-like capillary plasmas).
Balance equations for energy levels and ion stages for Crookes or Crookes-like plasma

Balance equations for the ion stages
(with $T_e(r,t)$, $T_i(r,t)$, $n_e(r,t)$, $V_e(r,t)$, $n(r,t)_i$ and $V_i(r,t)$ calculated by MHD)

$$\frac{dn_{\text{ion}}}{dt} = n_e \cdot \left\{ n_{\text{ion}}^{\frac{Z-1}{Z}} I^{\frac{Z}{Z-1}} + n_{\text{ion}}^{\frac{Z+1}{Z}} \left( R_{rr}^{Z+1} + R_{dr}^{Z+1} + R_{cr}^{Z+1} \right) - n_{\text{ion}}^{\frac{Z}{Z}} I^{\frac{Z}{Z}} - n_{\text{ion}}^{\frac{Z+1}{Z}} \left( R_{rr}^{Z} + R_{dr}^{Z} + R_{cr}^{Z} \right) \right\}$$ (63)

Balance equations for the energy levels
(with $T_e(r,t)$, $T_i(r,t)$, $n_e(r,t)$, $V_e(r,t)$, $n(r,t)_i$ and $V_i(r,t)$ calculated by MHD)

$$\frac{d(n_{\text{ion}}^Z)_j}{dt} = -(n_{\text{ion}}^Z)_j \left[ \sum_{i<j} (A_{ji} + P_{ji}^d) + \sum_{i>j} P_{ji}^e \right] + \sum_{i>j} (n_{\text{ion}}^Z)_i \left( A_{ji} + P_{ji}^d \right) + \sum_{i<j} (n_{\text{ion}}^Z)_i P_{ij}^e$$ (42)

$$A_{ji} \rightarrow E(\tau)A_{ji},$$

$$E(\tau) = \frac{1}{2} \int_{-X}^{X} \left[ \int_{0}^{1} \exp\left[ -\left( x - y \beta \tau \right)^2 \right] \left[ \exp\left( -\frac{1}{y} \int_{0}^{2\tau} \exp\left[ -\left( x - y \beta z \right)^2 \right] dz \right) \right] dy \right] dx$$ (44)

Spectra

l-ionization rate coefficients, R- recombination rate coefficients, A-Einstein coefficient, E(τ)- photon escape factor, P-electron excitation /de-excitation coefficients
Comparison of the x-ray (XUV) radiation from Crookes tube with Crookes-like capillary discharge

**Crookes tube:** It was found in the past that the electron beam produces low-intensity x-rays that make foggy marks on nearby unexposed photographic plates.

**Crookes-like capillary discharge:** High-intensity (~10 W at ~ 5 kHz) of x-rays (XUV) is easily achieved. The XUV nano-photolithography requires intensities more than about 50 W.
The x-ray (XUV) resonant line emission from the Crookes-like Oxygen capillary discharge

Fig. 12 X-ray (XUV) resonant line emission from the Crookes-like Oxygen capillary discharge (from http://www.google.com/patents/US6031241)
Intensity of x-ray (XUV) radiation vs Oxygen pressure of Crookes-like capillary discharge

Fig. 13 Intensity of x-ray (XUV) radiation vs Oxygen pressure of the Crookes-like discharge ($I_{\text{max}}=6\text{kA}$) (from http://www.google.com/patents/US6031241)
X-ray (XUV) resonant line emission from Crookes-like Xenon capillary discharge

Fig. 14 X-ray (XUV) resonant line emission from the Crookes-like Xenon capillary discharge (from http://www.google.com/patents/US6031241)
The use of Van Cittert-Zernike theorem for x-rays generated by Crookes-like plasma

Output aperture of an x-ray source based on a Crookes-like plasma

The Crookes-like plasma produces partially coherent or fully transversally coherent x-rays (waves) in the region.

For a circular (radius R) surface of incoherent (uncorrelated) emitters of a Crookes-like-ray source, we have

\[ |\mu_{12}| = \frac{2J_1(\langle k \rangle R\Theta)}{\langle k \rangle R\Theta} \] (69)

Thus the use of the Van Cittert-Zernike theorem for a Crookes-like x-ray source yields

Transverse coherence

- Incoherent radiation: \( 2R >> \langle \lambda \rangle Z / d \) (70)
- Partially coherent radiation: \( 2R \sim \langle \lambda \rangle Z / d \) (71)
- Coherent radiation: \( 2R << \langle \lambda \rangle Z / d \) (72)
Understanding Crookes-like capillary plasma sources requires theory, computations and experiments.

For an example:
T. Donkó, P. Hartmann, K. Kutasi, On the reliability of low-pressure DC glow discharge modelling, XXVIIthe ICPIG, Eindhoven, the Netherlands, 18-22 July (2005)

Why can the 25-year theoretical and experimental experience of the University of Pecs in capillary plasmas be useful for R&D of x-ray (XUV) sources for photolithography?

Fig. 16 Understanding Crookes-like plasma x-ray sources requires theory, computations and experiments.
Problems as home assignments (A)

1. Explain the transition of matter in the 1st state (solid-state) to matter in the 4th state.
2. Describe kinds of plasmas and basic processes in plasmas.
3. Calculate the electron plasma frequency, the Debye distance and Larmor (cyclotron) frequency for a LIP of the electron density $10^{15}$ cm$^{-3}$, kinetic temperature 50 eV and magnetic flux density 2 Tesla.
4. Show the connection of ionization and the Townsend model with the operation of a Crookes X-tube.
5. Describe the two types of emission of x-rays by Crookes-tube glow plasma.
6. Give a qualitative explanation of the continuum radiation by using Maxwell equations and the line emission by using atomic physics.
7. How does ionization "bottlenecks" limit the number of ionization states in a Crookes Ar-plasma?
8. Compare the microscopic model vs kinetic descriptions of plasma of a Crookes x-ray tube.
9. Show the transition of the Vlasov kinetic equation to the collisionless Maxwell-Vlasov equations for a Crookes glow-plasma.
10. Describe the basic points of the magnetohydrodynamic (MHD, fluid) model of a Crookes glow-plasma.
11. What are the two approaches in the MHD model of a Crookes glow-plasma.
12. Derive the continuity equation for mass (particles) conservation for MHD of a Crookes glow-plasma.
13. Derive the force equation for MHD of a Crookes glow-plasma.
Problems as home assignments (B)

14. Derive the conservation of energy equation for MHD of a Crookes glow-plasma.
15. Summarize MHD equations for a Crookes glow-plasma.
16. Give an example of 2 temperature ($T_e$ and $T_i$), 2 fluids ($N_e V_e$ and $N_i V_i$) MHD equations for a Crookes plasma.
17. Present balance equations for the energy levels and ion stages for a Crookes glow-plasma. Give explanations of the terms in the equations.
18. Why does understanding Crookes glow-plasma x-ray sources require theory, computations and experiments?
References

1. Abonyi Iván, A negyedik halmazállapot. Bevezető a plazmafizikába, Gondolat kiadó, Budapest (1971)

For additional information see: