Lecture 3

Interference, diffraction, coherence, refraction, reflection and absorption do affect x-ray lasers

Outline

- Basic wave phenomena involved in the operation and application of x-ray lasers (sources)
- From Wave equation to interference of x-rays. From Wave equation to diffraction of x-rays. Weakest, diffraction-less and strong diffractions of x-rays. Longitudinal (temporal) and transverse (spatial) coherences of x-rays by statistic optics. Longitudinal coherence-length $\ell_c$ is determined by the line spectral bandwidth $\Delta \lambda$. From the Fresnel integral to degree of transversal coherence of x-rays (for simplicity, 1-D case). Van Cittert-Zernike theorem for 2-D x-ray source. The use of Van Cittert-Zernike theorem for an incoherent circular x-ray source. The use of Van Cittert-Zernike theorem for passive and active mediums of x-ray sources.
- Manipulation of x-rays (waves) by refraction, reflection and absorption. From refraction index $n(\omega)$ to refraction, reflection and absorption of x-rays.
- From Maxwell equations to Wave equation and then to refractive index $n(\omega)$. The velocity $v_i$ of an oscillating electron of an atom driven by an incident x-ray. The current density $J_T$ in Wave Equation yields refractive index $n(w)$. Refractive index $n(w)$ in x-ray spectral region. Different forms of $n(w)$ in x-ray spectral region.
Outline ctd.

- Atomic scattering factors $f_0^1$ and $f_0^2$ for Carbon and Aluminum (examples). Refractive index $n(\omega)$ from the IR to X-ray region. Snell’s law for the total (100%) external reflection of x-rays at an interface. Total external reflection of x-rays and the glancing x-ray optics. From [Wave equation + refractive index $n(\omega)$] to Fresnel reflection coefficient $R$. From refraction index $n(\omega)$ to Fresnel reflection and refraction of x-rays at an interface. Reflection coefficient $R_s$ for the S-polarization Fresnel reflection coefficient $R_s$ at normal incidence. Fresnel $R_s$ at the glancing incidence $Q \sim Q_c$. The values $R_s$ at the glancing incidence $Q \sim Q_c$. Reflection coefficient $R_p$ for P-polarization.

- Modification of x-rays (waves) by reflection at glancing incidence. The guiding (focusing) by the Kirkpatrick-Baez glancing incidence, curved-mirror x-ray optics. Guiding and focusing by the capillary x-ray optics. “Focusing” x-ray femtosecond pulses by a tapered capillary on a nanometer scale. The free-space virtual source of a system of the “fs-nm focusing” of x-rays. The model shows feasibility of the “fs-nm focusing” of x-rays. Guiding and focusing by Kumakhov’s “lens”. Focusing of incoherent and coherent x-rays by a “Kumakhov’s lens”.

- The model results. Guiding and focusing of x-rays by a gradient refraction index plasma-based guide. Summary of guiding and focusing of x-rays by a gradient refraction index plasma-based guide. Guiding and focusing of x-rays by a plasma-based guide: transient modes $m$ ($m=0,1,2 \ldots$). The use of Van Cittert-Zernike theorem for the x-rays focused by a plasma-based guide.

- Problems as home assignments

- References

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Basic wave phenomena involved in the operation and application of x-ray lasers (sources)

In the following lectures 4-12, we will consider how interference, diffraction, coherence, refraction, reflection and absorption of x-rays affect operation of x-ray sources (lasers), manipulation of x-rays and their application.

Fig. 1 Interference, diffraction, coherence, refraction, reflection and absorption of x-rays are involved in the operation of x-ray sources, manipulation of x-rays and their applications.

Most equations related to interference, diffraction, coherence, refraction, reflection and absorption of x-rays are described by equations of conventional optics in the visible spectral region, but some of them require additional consideration !!!
From Wave equation to interference of x-rays

From the questions (What is x-ray radiation? Is it an EM wave or a ray?) to the interference of x-rays.

Fig. 2 A source of incoherent or coherent EM waves of x-rays

Maxwell’s equations \[\rightarrow\] Wave equation \[\rightarrow\] Interference of x-rays

Wave equation yields the principle of superposition (Fourier-decomposition) for EM waves. That is to say EM waves are given by

\[ E(r, t) = \sum_i E_i(r, t) \] (1)

\[ E_i(r, t) \sim e_i a_{Ti} (t - r_i/c + \phi_j)/r_j \sim e_i a_{Ti} (k_i r_i - \omega_i t + \phi_j)/r_j, \] (2)

where

\[ \omega = kc = (2\pi/\lambda_{x-ray})c, \] and the field

\[ E(r, t) \sim \sum_i e_i a_{Ti} (k_i r_i - \omega_i t + \phi_j)/r_j \] (3)

is induced by incoherent \((\phi_i \neq \phi_j, a_{Ti} \neq a_{Tj}, r_i \neq r_j)\) or coherent \((\phi_i = \phi_j, a_{Ti} = a_{Tj}, r_i \neq r_j\) or \(r_i = r_j\)) accelerated and/or de-accelerated \(|a_{Ti}| \neq 0\) electrons of the x-ray medium.

Intensity \((I)\) of the Fourier-like field \(E(r, t) \sim \sum_i e_i a_{Ti} (k_i r_i - \omega_i t + \phi_j)/r_j\) is given by

\[ I \sim |E|^2 = \sum_i E_i^2 + \sum_i \sum_j E_i E_j \] (4)

\[ E_i E_j > 0 \text{ – constructive interference} \]
\[ E_i E_j < 0 \text{ – destructive interference} \]

Interference term
From Wave equation to diffraction of x-rays

Maxwell’s equations $\rightarrow$ Wave equation $\rightarrow$ Diffraction of x-rays

Wave equations:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) E_T(r, t) = \frac{1}{\varepsilon_0} \frac{\partial J_T(r, t)}{\partial t}$$  \hspace{1cm} (6)

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \nabla^2\right) E_T(r, t) = 0$$  \hspace{1cm} (7)

Monochromatic x-rays:

$$E_T(r, t) = E_T(r)e^{i(\omega t)}$$, \hspace{0.5cm} \color{red}{\omega = kc = (2\pi/\lambda_{\text{x-ray}})c}  \hspace{1cm} (8)

Helmholtz differential equation:

$$\nabla^2 E_T(r) + k^2 E_T(r) = 0$$  \hspace{1cm} (9)

Integral form of Helmholtz differential equation:

$$\vec{E}_T(r) = \int\int_{\text{apert.}} \vec{E}_T^{(\text{inc}.)}(r') \frac{e^{ik|r-r'|}}{4\pi|r-r'|} \, dx' \, dy'$$  \hspace{1cm} (10)

Fresnel-Kirchhoff diffraction integral = secondary spherical waves, Huygens-Fresnel principle [1]
**Weakest, diffraction-less and strong diffractions of x-rays**

We now get an answer to the questions (What is x-ray radiation? Is it an EM wave or a ray?).

**Quasi-monochromatic waves:** 
\[ \mathbf{E}_T(r,t) = \mathbf{E}_T(r) e^{-i\omega t}, \text{ where } \omega = <\omega>, \quad k = <k>, \quad \lambda = <\lambda> \]

Weakest diffraction (smallest \( b \)) if
\[ \mathbf{E}_{T(\text{inc.})}(r,t) = \mathbf{E}_T(r,t) \exp(-\omega t + \varphi(r,t)) = \text{const.} \]
\[ b \sim L_{\text{fap}}(\lambda/a) \]

"Diffraction-less" = (near-field) = (wave = x-ray) = (geometric optics) if
\[ b \sim a \]

Strongest diffraction if
\[ \mathbf{E}_{\text{inc}}(r,t) = \mathbf{E}_T(r,t) \exp(\omega t + \varphi(r,t)) \text{ is incoherent,} \]
respectively \( b \rightarrow \infty \)

\[ \Delta x \Delta p > \hbar \text{ - Heisenberg uncertainty} \]
\[ \Delta x \Delta k > 1 \]
**Longitudinal (temporal) and transverse (spatial) coherencies of x-rays by statistic optics**

**Mutual coherence factor**
\[ \Gamma_{12}(\tau) = \langle E_1(P_1,t+\tau)E^*_2(P_2,t) \rangle \]

Degree of **spatial coherence** (coherence factor, mutual intensity)
\[ J_{12}(P_1,P_2) = \Gamma_{12}(0) = \langle E_1(P_1,t)E^*_2(P_2,t) \rangle \]

**Normalized degree of spatial coherence** (coherence factor)
\[ \mu_{12} = \frac{\langle E_1(P_1,t)E^*_2(P_2,t) \rangle}{\langle |E_1(P_1,t)|^2 \rangle^{1/2} \langle |E_2(P_2,t)|^2 \rangle^{1/2}} \]

**Longitudinal (temporal) full-coherence length**
\[ \ell_c = \frac{\lambda^2}{\Delta\lambda} \] (15)

**Transverse (spatial) full-coherence width**
\[ \Theta = \frac{\lambda}{4\pi a} \] (16)
\[ D = L_{fap} \Theta \] (17)

**Simple explanation:**
The phase difference between any two waves (rays) in the longitudinal or transverse direction is less than \(~0.5\) period.
**Longitudinal coherence-length** \( l_c \) is determined by the line spectral bandwidth \( \Delta \lambda \)

![Diagram showing line spectral bandwidth and longitudinal coherence length](image)

Fig. 7 The line spectral bandwidth and the longitudinal coherence length by the phase shift

The phase difference between the two waves is equal to 0.5 period.

with wavelength \( \lambda \)

\[
\lambda_{coh} = N \lambda
\]  

(18)

with wavelength \( \lambda + \Delta \lambda \)

\[
\lambda_{coh} = (N - 0.5)(\lambda + \Delta \lambda)
\]  

(19)

\[\Rightarrow N = \frac{\lambda}{2\Delta \lambda}\]  

(20)

\[\Rightarrow l_c = \frac{\lambda^2}{\Delta \lambda}\]  

(21)
From Fresnel’s integral to degree of transversal coherence of x-rays (for simplicity, 1-D case)

Mutual intensity (see, p.6):
\[ J_{ij}(P_i, P_j) = \Gamma_{12}(0) = \langle E_1(P_1, t) E_2^*(P_2, t) \rangle \]
Calc. by Fresnel’s integral (11)

For 1D incoherent source:
\[ J_{ij}(P'', P'') = A_I(P'') \delta_{12}(|P'' - P''|) \]
\[ J_{ij}(x'', x'') = A_I(x'') \delta_{12}(|x'' - x''|) \]
\[ J_{ij}(x_i, x_j) = \int_{-h}^{h} \int_{-h}^{h} J(x'', x'') \exp[-i n_0 c^{-1}(\omega)(r_i'' - r_j'')] \frac{\chi(\Theta''_i)}{r_i'' \lambda} \frac{\chi(\Theta''_j)}{r_j'' \lambda} dx'' dx'' \] (24)

\[ J_{ij}(x''_i, x''_j) \sim \sqrt{I(x''_i)I(x''_j)} \left[ \frac{\sin(2\pi \sqrt{(\Delta x'')^2 / \langle \lambda \rangle})}{2 \pi \sqrt{(\Delta x'')^2 / \langle \lambda \rangle}} \right] \]

The value in the box is normalized degree of spatial coherence (coherence factor)
\[ \mu_{ij} = \mu_{ij}(x'', x'') \] (25(a))
Van Cittert-Zernike theorem for 2-D x-ray source

\[ \mu_{12}(x_1, y_1; x_2, y_2) = e^{-i\psi} \int \int I(\xi, \eta) \exp \left[ i(k)(\xi \Delta x / z + \eta \Delta y / z) \right] d\xi d\eta / \int \int I(\xi, \eta) d\xi d\eta \]  

**Van Cittert-Zernike theorem:** \( \mu_{12} \) is the Fourier transform of the Intensity distribution of incoherent emitters (\( \psi \) is a phase parameter that is irrelevant for \( |\mu_{12}| \))

For circular (radius R) surface of incoherent (uncorrelated) emitters

\[ I(\xi, \eta) = I_0 \text{circ} \left( \sqrt{\xi^2 + \eta^2} / R \right) \]  

\[ \mu_{12} = \frac{\int I(\rho) J_0(\langle k \rangle \rho R) \rho d\rho}{\int_0^\infty I(\rho) \rho d\rho} = e^{-i\psi} \frac{2J_1(\langle k \rangle R \Theta)}{\langle k \rangle R \Theta} \]  

**Fig. 10** Degree of spatial coherence of the circular x-ray source \( |\mu_{12}(<k>R \Theta)| \)
The use of Van Cittert-Zernike theorem for an incoherent circular x-ray source

Fig. 11 Schema for the use of theorem in case of a 2-Dimensional (circular, radius R) source of incoherent, uncorrelated emitters

An incoherent x-ray source produces partially coherent or fully transversally coherent x-rays (waves) in the region

For a circular (radius R) surface of incoherent (uncorrelated) emitters

\[ |\mu_{12}| = \frac{2J_1(\langle k \rangle R\Theta)}{\langle k \rangle R\Theta} \]  

(29)

Thus the use of Van Cittert-Zernike theorem yields

Transverse coherence

- **Incoherent** radiation: \( 2R \gg \langle \lambda \rangle Z / d \)  
- **Partially coherent** radiation: \( 2R \sim \langle \lambda \rangle Z / d \)  
- **Coherent** radiation: \( 2R \ll \langle \lambda \rangle Z / d \)

(30)  
(31)  
(32)

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The use of Van Cittert-Zernike theorem for passive and active mediums of x-ray sources

From Van Cittert-Zernike theorem:
- Incoherent radiation: $2R \gg \lambda \frac{L_{fap}}{d}$
- Partially coherent radiation: $2R \sim \lambda \frac{L_{fap}}{d}$
- Coherent radiation: $2R \ll \lambda \frac{L_{fap}}{d}$

Fig. 12 Incoherent, partially coherent and coherent X-rays from incoherent source

From Van Cittert-Zernike theorem:
- Incoherent radiation: $2R \gg \lambda \frac{L}{R}$
- Partially coherent radiation: $2R \sim \lambda \frac{L}{R}$
- Coherent radiation: $2R \ll \lambda \frac{L}{R}$

Fig. 13 Incoherent, partially coherent and coherent x-rays from the passive and active mediums of x-ray sources
Manipulation of x-rays (waves) by refraction, reflection and absorption

Manipulation of x-rays by refraction, reflection and absorption

Optical models and formulas for the x-ray spectral region?

Lenses, mirrors, gratings, attenuators, filters and waveguides for incoherent and/or coherent x-rays?
From refraction index $n(\omega)$ to refraction, reflection and absorption of x-rays

**Fig. 14** Refraction and reflection of an incident wave of x-rays

**Fig. 15** Absorption (attenuation) of an incident wave of x-rays

$$I(\lambda) = I_0(\lambda) e^{-\rho \mu L} ; \mu = \mu(n_2)$$
From Maxwell equations to Wave equation and then to refractive index \( n(\omega) \)

Maxwell equations \( \rightarrow \) Transverse Wave Equation

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) E_T (r, t) = -\frac{1}{\varepsilon_0} \frac{\partial J_T (r, t)}{\partial t}
\]

(33)

In the case of the refraction, reflection and absorption of x-rays, the positions of electrons in atoms and the positions of electrons are irrelevant under the forward scattering of x-rays by the atom mediums. Then, for an atom with \( Z \) electrons, we get

\[
J_T (r, t) = -en_{at} \sum_i Z g_i v_i (r, t)
\]

(34)

where

\[
Z = \sum_i g_i
\]

(35)

Then we should find the velocity \( v_i (r, t) \).

\[
v_i (r, t) ?
\]

(36)
**The velocity** $\mathbf{v}_i$ of an oscillating electron of an atom driven by an incident x-ray

We use the **Lorentz classical model** of an oscillating bound electron of an atom. The EM force of the incident wave of x-rays changes the electron coordinate ($\mathbf{r}$) and velocity ($\mathbf{v}(\mathbf{r},t) = \frac{d\mathbf{r}}{dt}$) and acceleration ($\mathbf{a}(\mathbf{r},t) = \frac{d^2\mathbf{r}}{dt^2}$):

$$m \frac{d^2\mathbf{r}}{dt^2} + m\gamma \frac{d\mathbf{r}}{dt} + m\omega_i^2 \mathbf{r} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$  \hspace{1cm} (37)

In the case of time-harmonic incident wave with the field $\mathbf{E}$ given by

$$\mathbf{E} = \mathbf{E}_0 e^{-i\omega t}$$  \hspace{1cm} (38)

we get a solution

$$\mathbf{r}(t) = \frac{1}{\omega^2 - \omega_i^2 + i\gamma\omega} \frac{e\mathbf{E}_0 e^{-i\omega t}}{m}$$  \hspace{1cm} (39)

which yields

$$\mathbf{v}_i(t) = \frac{e}{m} \frac{1}{\omega^2 - \omega_i^2 + i\gamma\omega} \frac{d\mathbf{E}}{dt}$$  \hspace{1cm} (40)
Current density \( J_T \) in Wave equation yields refractive index \( n(\omega) \)

Thus Eqs. (34) and (40) yield

\[
J_T (r, t) = -\frac{e^2 n_{at}}{m} \sum_i \frac{g_i}{\omega^2 - \omega_i^2 + i\gamma \omega} \frac{dE_T}{dt} \quad (41)
\]

The use of Eq. (40) in Eq. (33) yields the equation

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right) E_T (r, t) = \frac{e^2 n_{at}}{\varepsilon_0 m} \sum_i \frac{g_i}{\omega^2 - \omega_i^2 + i\gamma \omega} \frac{d^2 E_T}{dt^2} \quad (42)
\]

and the equation

\[
\left[ \left( 1 - \frac{e^2 n_{at}}{\varepsilon_0 m} \sum_i \frac{g_i}{\omega^2 - \omega_i^2 + i\gamma \omega} \right) \frac{\partial^2}{\partial t^2} - c^2 \nabla^2 \right] E_T (r, t) = 0 \quad (43)
\]

which can be presented in the conventional form of the Wave equation

\[
\left[ \frac{\partial^2}{\partial t^2} - \frac{c^2}{n^2(\omega)} \nabla^2 \right] E_T (r, t) = 0 \quad (44)
\]
Refractive index $n(\omega)$ in x-ray spectral region

The comparison Eqs. (43) and (44) yields the refraction index

$$n(\omega) = \left( 1 - \frac{e^2 n_{at}}{\varepsilon_0 m} \sum_i \frac{g_i}{\omega^2 - \omega_i^2 + i\gamma \omega} \right)^{1/2}$$  \hspace{1cm} (45)

In the case of x-rays, $\omega >> e^2 n_{at}/\varepsilon_0 m$. That yields the approximation

$$n(\omega) = 1 - \frac{1}{2} \frac{e^2 n_{at}}{\varepsilon_0 m} \sum_i \frac{g_i}{\omega^2 - \omega_i^2 + i\gamma \omega}$$  \hspace{1cm} (46)

If $\omega > \omega_i$, then $n(\omega) < 1$ and the phase velocity $v = c/n > c$. \hspace{1cm} (47)

If $\omega < \omega_i^2$, then $n(\omega) > 1$ and the phase velocity $v = c/n < c$. \hspace{1cm} (48)
Different forms of $n(\omega)$ in x-ray spectral region

Refractive index $n(\omega)$ is usually represented by using atomic real ($f_{01}^0$) and imaginary ($f_{02}^0$) scattering factors or the respective real ($\delta$) and imaginary ($\beta$) parameters. For the representation of

$$n(\omega) = 1 - \frac{1}{2} \frac{e^2 n_{at}}{\varepsilon_0 m} \sum_i g_i \frac{1}{\omega^2 - \omega_i^2 + i \gamma \omega}$$  \hspace{1cm} (49)$$

we note that for forward wave scattering, the scattering factor is given by

$$f^0(\omega) = f_{1}^0(\omega) - if_{2}^0(\omega) = \sum_i g_i \omega^2 \frac{1}{\omega^2 - \omega_i^2 + i \gamma \omega}$$  \hspace{1cm} (50)$$

that yields

$$n(\omega) = 1 - n_{at} r_e \frac{\lambda^2}{2\pi} \left[ f_{1}^0(\omega) - if_{2}^0(\omega) \right]$$  \hspace{1cm} (51)$$

where

$$r_e = \frac{e^2}{4\pi \varepsilon_0 mc^2}$$

is the classical electron radius. To this end, the definition

$$\delta = \frac{n_{at} r_e \lambda^2}{2\pi} f_{1}^0(\omega)$$  \hspace{1cm} (52)$$

$$\beta = \frac{n_{at} r_e \lambda^2}{2\pi} f_{2}^0(\omega)$$  \hspace{1cm} (53)$$

yields the conventional form used in the x-ray spectral region and the research fields related to x-ray lasers

$$n(\omega) = 1 - \delta + i\beta$$  \hspace{1cm} (54)$$
Atomic scattering factors $f_0^1$ and $f_0^2$ for Carbon and Aluminum (examples)

Fig. 16 Scattering factors of C and Al (from B.L. Henke, E.M. Gullikson, J.C. Davis, *Atomic Dat. And Nucl. Dat. Tables* 54, 181 (1993))
Refractive index $n(\omega)$ from the IR to X-ray region

A simple analysis yields the schematic dependence $n=n(\omega)$ from IR to X-ray region.

**Fig. 17** Schematic behavior of the refractive index vs frequency from the IR to X-ray spectral region.
Snell’s law for the total (100%) external reflection of x-rays at an interface

Most equations in the x-ray optics are described by equations of conventional optics in the visible spectral region, but total reflection of x-rays at an interface requires additional consideration!

**Snell’s law:** \[
\frac{\sin \varphi''}{\sin \varphi} = \frac{n_1}{n_2} \tag{55}
\]

In the visible region: \( n_1(\lambda) > 1, \quad n_2(\lambda) = 1 \tag{56} \)

In the x-ray region: \( n_1(\lambda) = 1, \quad n_2(\lambda) = 1 - \delta + \beta \sim 1 - \delta < 1 \tag{57} \)

**Fig. 18** Total external reflection of the waves of x-rays

The visible region: (from (56)) \( \rightarrow \) Total internal reflection

The x-ray region: (from (57)) \( \rightarrow \) Total external reflection

**Total reflection at** \( \varphi'' = \pi/2 \)

\( \Theta = \pi/2 - \varphi \)
Total external reflection of x-rays and glancing x-ray optics

Total reflection at $\varphi_c''=\pi/2$

$\Theta = \pi/2 - \varphi$

$\frac{1}{1 - \frac{1}{2} \Theta_c^2 + \ldots} = \frac{1}{1 - \delta}$

$\Theta_c = \sqrt{2\delta}$ (58)

Using $\delta = \frac{n_{at.} r_e \lambda^2}{2\pi} f_1^0(\omega)$ in Eq. (58) yields

$\Theta_c = \sqrt{\frac{n_{at.} r_e \lambda^2}{\pi} f_1^0(\omega)} \sim \lambda \sqrt{Z}$ (59)

Glancing incidence x-ray optics

Fig. 19 Glancing Incidence ($\Theta<\Theta_c$) and total reflection of x-rays

Fig. 20 Glancing Incidence x-ray optics (Kirkpatrick-Baez and Kumakhov optics)

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From “Wave equation + refractive index \( n(\omega) \)”

to Fresnel reflection coefficient \( R \)

From Snell’s law: \( R = 1, \) if \( \Theta < \Theta_c = \sqrt{2}\delta \)
\[ R = R\Theta \]?

Fig. 21 Refraction and reflection of an incident wave of x-rays

Wave Equation:
\[
E_{\text{inc.}} = E = E_0 e^{-i(\mathbf{k}r - \omega t)}\]  
\[
E_{\text{reflect.}} = E' = E_0' e^{-i(\mathbf{k}r' - \omega' t)}\]  
\[
E_{\text{refract.}} = E'' = E_0'' e^{-i(\mathbf{k}'r'' - \omega'' t)}\]  

Boundary conditions
Dirichlet: \( f^{(1)}(z=0) = f^{(2)}(z=0) \)
Neumann: \( df^{(1)}(z=0)/dz = df^{(2)}(z=0)/dz \)

reflection coefficient \( R \)
From refraction index \( n(\omega) \) to Fresnel reflection and refraction of x-rays at an interface

**Linear processes**

\[
\hbar \omega = \hbar \omega' = \hbar \omega'' \quad (65)
\]
\[
k = k' = \omega / c \quad (66)
\]
\[
k'' = \frac{\omega}{c/n_2} = \frac{\omega}{c} \left(1 - \delta + i\beta\right) \quad (67)
\]

**Wave equation**

\[
E_{\text{inc.}} = E = E_0 e^{-i(kr_{\omega t})} \quad (68)
\]
\[
E_{\text{reflect.}} = E' = E'_0 e^{-i(kr_{\omega t})} \quad (69)
\]
\[
E_{\text{refract.}} = E'' = E''_0 e^{-i(kr'_{\omega t})} \quad (70)
\]

**Continuity of \( E \) and \( H \) components which are parallel to the interface**

\[
\left(\frac{z}{|z|}\right) \times (E_0 + E_0') = \left(\frac{z}{|z|}\right) \times E_0'' \quad (71)
\]
\[
\left(\frac{z}{|z|}\right) \times (H_0 + H_0') = \left(\frac{z}{|z|}\right) \times H_0'' \quad (72)
\]

**Continuity of the \( D \) and \( B \) components which are parallel to the interface**

\[
\left(\frac{z}{|z|}\right) \times (D_0 + D_0') = \left(\frac{z}{|z|}\right) \times D_0'' \quad (73)
\]
\[
\left(\frac{z}{|z|}\right) \times (B_0 + D_0') = \left(\frac{z}{|z|}\right) \times B_0'' \quad (74)
\]

\[k \sin \varphi = k' \sin \varphi' = k'' \sin \varphi'\]

**Incidence angle = Reflection angle**

\[\varphi = \varphi'\]

**Snell's law:**

\[
\frac{\sin \varphi''}{\sin \varphi} = \frac{1}{n_2} \quad (77)
\]
Reflection coefficient $R_s$ for S-polarization

**S-polarization:** E is perpendicular to the plane of incidence

Continuity of tangential components of $E$ and $H$ and Eq. (40) from Lecture 2 yields

For reflected x-rays

$$
\frac{E_0'}{E_0} = \frac{\cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}}{\cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}}
$$

(78)

For refracted x-rays

$$
\frac{E_0''}{E_0} = \frac{2 \cos \varphi}{\cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}}
$$

(79)

Then Fresnel reflection coefficient $R_s$ is given by

$$
R_s = \frac{\langle I' \rangle}{\langle I_0 \rangle} = \frac{\langle S' \rangle}{\langle S_0 \rangle} = \frac{(1/2) \text{Re}(E_0' \times H_0')}{(1/2) \text{Re}(E_0 \times H_0)} = \frac{|E_0'|^2}{|E_0|^2}
$$

(80)

$$
R_s = \left| \frac{\cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}}{\cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}} \right|^2
$$

(81)
Fresnel reflection coefficient $R_s$ at normal incidence

S-polarization and normal incidence: $\varphi = 0$

\[
R_s = \frac{\left|\cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}\right|^2}{\left|\cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}\right|^2} = \frac{1 - n_2^2}{1 + n_2^2}
\]  

(81)

(82)

The refraction index is given by Eq. (54) as

\[
n_2 = 1 - \delta + i\beta
\]  

(83)

The use of (83) in Eq. (82) yields

\[
R_s = \left(\frac{1 - n_2}{1 + n_2}\right)^2 = \frac{\delta^2 + \beta^2}{(2 - \delta)^2 + \beta^2}
\]  

(84)

For $\delta \ll 1$ and $\beta \ll 1$, we finally get

\[
R_s = \frac{\delta^2 + \beta^2}{4}
\]  

(85)

An example: Al at $\lambda = 4$ nm.

$f_0^1 = 12$ and $f_0^2 = 4.7$ (from B.L. Henke, E.M. Gullikson, J.C. Davis, Atomic Dat. And Nucl. Dat. Tables 54, 181 (1993)).

That yields $\delta \sim 10^{-2}$ and $\beta \sim 5 \times 10^{-3}$

$R_s \downarrow = 5 \times 10^{-5} \ll 1$ Mirrors for x-rays?
Fresnel $R_s$ at the glancing incidence $\Theta \sim \Theta_c$

S-polarization and glancing incidence: $\Theta < \Theta_c$

$$R_s = \frac{|\cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}|^2}{|\cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}|^2}$$

$$\Theta = \frac{\pi}{2} - \varphi < \left(\Theta_c = \sqrt{2\delta} \ll 1\right)$$

$$\cos \varphi = \sin \Theta \approx \Theta$$

$$\sin^2 \varphi = 1 - \cos^2 \varphi = 1 - \sin^2 \Theta \approx 1 - \Theta^2$$

$$\left(n_2\right)^2 = (1 - \delta + i\beta)^2 = (1 - \delta)^2 + 2i\beta(1 - \delta) - \beta^2$$

$$R_s(\Theta) = \frac{|\Theta - \sqrt{\Theta^2 - \Theta_c^2} + 2i\beta|^2}{|\Theta + \sqrt{\Theta^2 - \Theta_c^2} + 2i\beta|^2}$$

Fig. 22 Total reflection of x-rays at $\Theta < \Theta_c$

Fig. 23 $R_s$ calculated by (92) for glancing incidence $\Theta \sim \Theta_c$
Values $R_s$ at the glancing incidence $\Theta \sim \Theta_c$

Fig. 24 The dependences of $R_s$ for x-rays at the glancing ($\Theta \sim \Theta_c$) incidence for Carbon (from B.L. Henke, E.M. Gullikson, J.C. Davis, *Atomic Dat. And Nucl. Dat. Tables* 54, 181 (1993))

Fig. 25 The dependences of $R_s$ for x-rays at the glancing ($\Theta \sim \Theta_c$) incidence for Aluminum (from B.L. Henke, E.M. Gullikson, J.C. Davis, *Atomic Dat. And Nucl. Dat. Tables* 54, 181 (1993))
Reflection coefficient $R_p$ for P-polarization

**P-polarization:** $E$ is parallel to the plane of incidence

Continuity of tangential components of $E$ and $H$ and Eq. (40) from Lecture 2 yields

For reflected x-rays

$$E_0' = \frac{n_2^2 \cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}}{E_0}$$

For refracted x-rays

$$E_0'' = \frac{2 \cos \varphi}{n_2^2 \cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}}$$

Then Fresnel reflection coefficient $R_p$ is given by

$$R_p = \frac{\langle I' \rangle}{\langle I_0 \rangle} = \frac{\langle S' \rangle}{\langle S_0 \rangle} = \frac{(1/2) \text{Re}(E_0' \times H_0')}{(1/2) \text{Re}(E_0 \times H_0)} = \left| \frac{E_0'}{E_0} \right|^2$$

$$R_p = \frac{n_2^2 \cos \varphi - \sqrt{n_2^2 - \sin^2 \varphi}}{n_2^2 \cos \varphi + \sqrt{n_2^2 - \sin^2 \varphi}}^2$$

Notice, for normal incidence

$$R_p = R_s = \frac{\delta^2 + \beta^2}{4} \ll 1$$

Mirrors for x-rays?
Modification of x-rays (waves) by reflection at glancing incidence

Fig. 26 Modification of x-rays (waves), which are produced by an incoherent or coherent x-ray source, via the reflection of x-rays from a solid-state material at glancing incidence

Glancing incidence, curved-mirror x-ray optics?

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
The guiding (focusing) by the Kirkpatrick-Baez glancing incidence, curved-mirror x-ray optics

The Kirkpatrick-Baez glancing-incident, curved-mirror x-ray optics

Fig. 27 Guiding and focusing of x-rays by the Kirkpatrick-Baez glancing-incident, curved-mirror x-ray optics
Guiding and focusing by the capillary X-optics

Mono-guide (capillary): $R \sim 100\%$

- (a)
- (b)
- (c)

Poly-guides (capillaries): $R \sim 100\%$

- (d)

focusing "lens"?

Kumakhov's optics

(e)

collimating "lens"?

Kumakhov's optics

(f)

Fig. 28 Capillary optics [4]
“Focusing” x-ray femtosecond pulses by a tapered capillary on a nanometer scale

Femto-nano “focusing” of an x-ray ($\lambda=10\text{nm}$) pulse

Input x-ray fs-pulse

Output x-ray fs-pulse

$t$ $\Delta t \sim 20\text{fs}$

$t$ $\Delta t \sim 20\text{fs}$

$L_{\text{cap}}=10\text{cm}$

$\lambda=10\text{nm}$

$2b=20\text{ nm}$

$2a=20\text{ \mu m}$

$\text{SiO}_2$

Output aperture of x-ray source

$D$

$d$

$z$

Fig. 29 “Focusing” x-ray femtosecond pulses by a tapered capillary on a nanometer scale. The phenomenon was shown by S.V. Kukhlevsky et al., “Interference and diffraction in capillary x-ray optics”, X-Ray Spectr. 32: 223-228 (2003)

Importance of the wave phenomena at $L_{\text{cap}} > ab / \lambda$

(98)
Free-space virtual source of a system of the “fs-nm focusing” of x-rays

Fig. 30 A free-space virtual source of a system of the fs-nm focusing” of x-rays (from S.V. Kukhlevsky and G. Nyitray, Phys. Let. A, 291, 459 (2001))
The model shows feasibility of the “fs-nm focusing” of x-rays

Fig. 31 The model results demonstrate feasibility of the “fs-nm focusing” of x-rays with $\lambda=10$ nm. The phenomenon was shown by S.V. Kukhlevsky et al. in Phys. Let. A, 291, 459 (2001)

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
Guiding and focusing by Kumakhov’s “lens”

Kumakhov’s polycapillary “lens”

Computations

Fig. 32 Computation of the focusing of incoherent and coherent x-rays by a polycapillary lens (from S.V. Kukhlevsky, “Focusing of incoherent and coherent X-ray radiation by a polycapillary lens”, NURT-2006, Havana, Cuba, April 3-7, 2006)

Coherent x-ray sources: Synchrotrons of 3rd and 4th generation; FEL; Plasma-based lasers; HHG; Point-like laboratory and astrophysical plasma sources

Photo from M.A. Kumakhov

Fig. 33 (a) An original Kumakhof’s polycapillary ‘lens’. (b) A modern Kumakhov’s optics (from http://www.xos.com)

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
Focusing of incoherent and coherent x-rays by a Kumakhov’s “lens”: The model results

Non-coherent input x-rays: \(| \mu_{12}(P_1, P_2) | = 0 \\
E_{ph} = 0.1 \text{ keV}, \\
N_{ph} = 20,000,000 \)

Coherent input x-rays: \(| \mu_{12}(P_1, P_2) | = 1 \\
E_{ph} = 0.1 \text{ keV}, \\
N_{ph} = 20,000,000 \)

\(Z = 0.5 \text{ cm (focus)}\)

\(D_{FWHM} \approx 15 \mu m\) \(\rightarrow\) \(D_{FWHM} \approx 7 \mu m\)

**Fig. 34** The model results for the focusing of incoherent and partially coherent x-rays by a Kumakhov’s “lens”. The improvement was shown by S.V. Kukhlevsky et al. in S.V. Kukhlevsky, “Focusing of incoherent and coherent X-ray radiation by a polycapillary lens”, NURT-2006, Havana, Cuba, April 3-7, 2006: In the cooperation (L. Vincze, S.V. Kukhlevsky, K. Janssens), SPIE Vol. 5536, 81-85 (2004)]

TÁMOP-4.1.1.C-12/1/KONV-2012-0005 project
Guiding and focusing of x-rays by a gradient refraction index plasma-based guide

Fig. 35 Guiding and focusing of an x-ray beam by a gradient-index plasma guide (from S.V. Kukhlevsky, L. Kozma, Contributions to Plasma Physics 38: 583-597 (1998))

\[ n(r) = 1 - N_e(r)/2N_c, \quad (99) \]

\[ N_e(r) = N_e(0) + [N_e(C) - N_e(0)](r/C)^2 \quad \text{if} \quad -R(z) < r < R(z), \quad (100) \]

\[ = 0, \quad \text{if} \quad R(z) < r < -R(z), \quad (101) \]

\[ = 0, \quad \text{if} \quad L < z < 0, \quad (102) \]
Summary of guiding and focusing of x-rays by a gradient refraction index plasma-based guide

Within **MHD approximation**, WE yields

\[
\frac{\partial^2 \vec{E}}{\partial z^2} + \nabla_\perp^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = \frac{\omega_p^2}{c^2} \frac{N_e(r)}{N_e(0)} \vec{E} + \nabla (\nabla \cdot \vec{E}) \tag{103}
\]

The **ray-tracing approximation** of Eq. (103) yields

\[
\frac{d}{ds} \left( n \frac{d\vec{r}}{ds} \right) = \nabla n \tag{104}
\]

The **paraxial-envelope approximation** of Eq. (103) yields

\[
2ik \frac{\partial E}{\partial \zeta} = \nabla_\perp^2 E + \frac{\omega_p^2}{c^2} \left( \frac{N_e(r)}{N(0)} - 1 \right) E \tag{105}
\]

From Eq. (104), we get the ray trajectories in paraxial approximation (small angles) with:

\[
\frac{r}{C/\gamma} = r(0) \cos \left( \frac{z}{C/\gamma} \right) + \frac{1}{n(r)} \frac{dr(0)}{dz} \sin \left( \frac{z}{C/\gamma} \right) \tag{106}
\]

\[
\gamma = \sqrt{2(N_e(C) - N_e(0))/(2N_c - N_e(0))} \tag{107}
\]

From Eq. (106), we see that the beam intensity distribution is periodically reproduced with period \(z_p\):

\[
P(r, z - m z_p, \varphi) - P(r, z, \varphi) \tag{108}
\]

The plasma guide of length \(L\) supports transient mode \(m\) \((m = 0, 1, 2, \ldots)\) if

\[
\frac{L}{z_p} = \frac{L}{\sqrt{2\pi C} \cdot \sqrt{\frac{N_e(C) - N_e(0)}{(2\pi/r_e \lambda^2) - N_e(0)}}} = m \tag{109}
\]

The plasma does guide x-rays if the value \(L/z_p > 0.1\), which yields condition

\[
\Delta N_e^{\text{min}} = \frac{0.2 \pi^2 C^2}{L^2} \cdot [2\pi/r_e \lambda^2 - N_e(0)] \tag{110}
\]

(Transient modes in a plasma waveguide were introduced by S.V. Kukhlevsky et al. in Contr. Plasma Phys. 38: 583-597 (1998))
Guiding and focusing of x-rays by a plasma-based guide: transient modes m (m=0,1,2 ...)
The use of Van Cittert-Zernike theorem for x-rays focused by a plasma-based guide

Output aperture of an x-ray source based on plasma guide

**Fig. 32** Schema for the use of theorem in case of an x-ray source based on plasma guide (2-Dimensional, circular, radius $R$) source composed from incoherent (uncorrelated) emitters

Plasma guide produces partially coherent or fully transversally coherent x-rays (waves) in the region.

For a circular (radius $R$) surface of incoherent (uncorrelated) emitters of a plasma-guide, we have

$$|\mu_{12}| = \frac{2J_1(k R \Theta)}{\langle k \rangle R \Theta}$$

(111)

Thus the use of the Van Cittert-Zernike theorem for an x-ray source based on plasma-guide yields

**Transverse coherence**

- **Incoherent** radiation: $2R \gg <\lambda>Z / d$
- **Partially coherent** radiation: $2R \sim <\lambda>Z / d$
- **Coherent** radiation: $2R \ll <\lambda>Z / d$

(112)  (113)  (114)
Problems as home assignments (A)

1. Show that the principle of superposition (Fourier-like decomposition) for EM waves does not contradict wave equation.
2. How does the principle of superposition for EM waves yield interference phenomenon?
3. How does Fresnel-Kirchhoff diffraction integral relate to wave equation, Helmholtz equation and the secondary spherical waves of the Huygens-Fresnel principle?
4. Explain the weakest, “diffraction-less” and the strongest forms of diffractions.
5. What are the relations between the mutual coherence factor, the degree of spatial coherence (coherence factor, mutual intensity) and the normalized degree of spatial coherence (coherence factor)?
6. Calculate the longitudinal coherence-length for the spontaneous transition determined by the Einstein coefficient $A_{ul} = 1/\Delta\tau_{ul} = 10^{10} \text{s}^{-1}$.
7. For a point source having the intensity distribution $I(\rho) = I_0\delta(\rho)/2\pi\rho$ demonstrate that the radiation is fully coherent with $|\mu_{12}|=1$.
8. For an incoherent source having the Gauss intensity distribution $I(\rho) = I_0\exp(-\rho^2 / 2a^2$ demonstrate that the value $|\mu_{12}|=0.88$.
9. Show the difference between applications of the van Cittert-Zernike theorem for passive and active mediums.
10. Use the van Cittert-Cernik theorem and the three forms of diffraction for the explanation of the angle divergence of the incoherent x-ray source of Fig. 5 (Lecture 2) radiating the soft X-rays with the wavelength $\lambda \ll r_T$ if the aperture dimension $r_T = 0.1 \text{ mm}$ and $L_{ap} = 0.5 \text{ m}$.
11. For the incoherent source of Fig. 5 (Lecture 2) radiating the soft x-rays with the wavelength $\lambda = 50 \text{ nm}$ find the aperture dimension $r_T$, when the incoherent source will be transversally coherent (let $L_{ap} = 0.5 \text{ m}$). Explain the result by using the van Cittert-Cernik theorem and the three forms of diffraction.
Problems as home assignments (B)

12. For the source of coherently accelerated electrons of Fig. 7 (Lecture 2) radiating the soft x-rays with the wavelength $\lambda = 50\text{nm}$ find the aperture dimension $r_T$ that provides the transversally coherent x-rays (let $L = 0.5\text{m}$). Use the van Cittert-Cernike theorem and the one of the three forms of diffraction.
13. How are the x-rays (waves) modified by refraction, reflection and absorption?
14. Show relations between the refraction index $n$ and refraction, reflection and absorption. How do Maxwell’s equations and wave equation explain the refractive index $(n)$?
15. What is the connection between the Lorentz model and the refraction index?
16. Show the role of the atomic scattering factors $f_0$, $f_1$, and $f_2$.
17. Explain the variation of the refractive index from IR to the x-ray spectral region.
18. Explain the difference between total internal and external reflections.
20. What is the difference between the Kirkpatrick-Baez and Kumakhov glancing x-ray optic?
21. Discuss the “focusing” x-ray femtosecond pulses by a tapered capillary on a nanometer scale.
22. Explain the guiding and focusing of x-rays by a gradient refraction index plasma-based guide.
References


For additional information see: